2.3 Volume

Focus on …

- solving problems involving the volume of three-dimensional objects
- finding an unknown dimension of a three-dimensional object given its volume

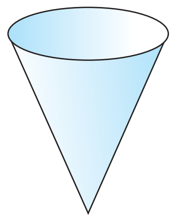
Western Canada is rich in natural resources. To make the most of Canada's natural resources, citizens need to understand the mathematics required to determine what they have, how to access it, and what it is worth. Whether it is recovering diamonds from Nunavut and the Northwest Territories or potash from Saskatchewan, transporting oil, or harvesting wood, measurement is a critical part of responsibly using Canada's resources. For many resources, volume is an important measurement: barrels of oil, board feet of lumber, and bushels of grain are just some of the traditional units of volume.

Investigate Volume

Materials

- conical cup
- paper
- scissors
- tape or glue
- sand, rice, or popcorn

What is the relationship between the volume of a right cone and the volume of a right cylinder?



1. Using a right conical cup for reference, create a right cylinder with the same height and the same base area as the cone.
2. If you fill the cup with material (for example, rice, sand, or popcorn), predict how many cups of your material it will take to fill the cylinder.
3. Test your prediction.
4. **Reflect and Respond**
   a) Discuss with a partner the results of your investigation.
   b) Is there a relationship between the amount of material that the cone holds and the amount of material that the cylinder can hold? What is the relationship?

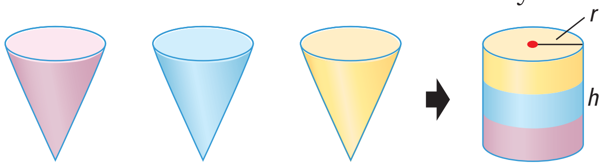
Link the Ideas

Volume of a Right Cone

The volume of a right prism or right cylinder can be found by multiplying the area of its base, B, by its height, h.

$V = Bh$

The volume of a right cone is related to the volume of a right cylinder with the same radius and height. The volume of the cone is one third of the volume of the cylinder.

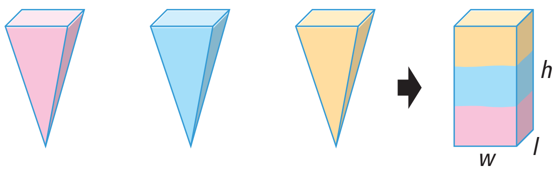


For a right cone with radius r and height h, $V_{\text{cone}} = \frac{1}{3}Bh$.

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

Volume of a Right Pyramid

This relationship is also true for right rectangular prisms and right pyramids. The volume of a right pyramid is one third of the volume of a right prism with the same base and height.



For a right pyramid with a rectangular base of length (l), width (w), and a perpendicular distance (h) from the base of the pyramid to its **apex**, $V_{\text{pyramid}} = \frac{1}{3}Bh$.

$$V_{\text{pyramid}} = \frac{1}{3}lwh$$

Volume of a Sphere

The volume of a sphere is two-thirds the volume of a cylinder with the same radius and a height equal to the diameter of the sphere. If the sphere has a radius r, then the cylinder has a base radius r and a height $2r$.

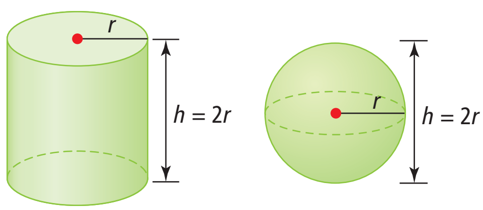
$V_{\text{sphere}} = \frac{2}{3}(\text{volume of cylinder})$

$V_{\text{sphere}} = \frac{2}{3}\pi r^2 h$

$V_{\text{sphere}} = \frac{2}{3}\pi(r^2)(2r)$

$V_{\text{sphere}} = \frac{4}{3}\pi r^3$

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$



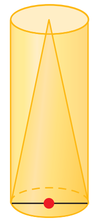
Web Link

To watch a video showing the relationship between the volume of a right cone and the volume of a right cylinder, go to www.mhrmath10.ca and follow the links.

apex

- the highest point of a pyramid

Example 1 Calculate the Volume of a Right Cylinder and a Right Cone



a) Calculate the volume of the right cylinder, to the nearest tenth of a cubic centimetre.

b) Calculate the volume of the right cone, to the nearest tenth of a cubic centimetre.

Solution

a) For the cylinder,

$r = 5 \div 2$ $\quad h = 12$

$\ \ = 2.5$

Substitute into the formula $V = \pi r^2 h$.

$V = \pi r^2 h$
$V = \pi(2.5)^2(12)$
$V = 75\pi$
$V = 235.619\ldots$

The exact value is 75π cm^3; an approximate value is 235.6 cm^3.

The volume of the cylinder is approximately 235.6 cm^3.

b) For the cone,

$r = 5 \div 2$ $\quad h = 12$

$\ \ = 2.5$

Method 1: Use a Formula

Substitute into the formula $V = \frac{1}{3}\pi r^2 h$.

$V = \frac{1}{3}\pi r^2 h$
$V = \frac{1}{3}\pi(2.5)^2(12)$
$V = 25\pi$
$V = 78.539\ldots$

The exact value is 25π cm^3, an approximate value is 78.5 cm^3.

The volume of the cone is approximately 78.5 cm^3.

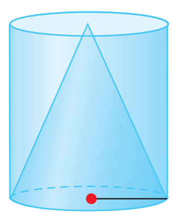
Method 2: Use Volume Relationships

Since the volume of the cone is one third of the volume of the cylinder, you could divide the volume of the cylinder by three.

$V = \frac{1}{3}(235.6)$
$V = 78.533\ldots$

The volume of the cone is approximately 78.5 cm^3.

Your Turn



a) What is the volume of the right cylinder, to the nearest cubic centimetre?

b) What is the volume of the right cone, to the nearest cubic centimetre?

Example 2 Calculate the Volume of a Right Pyramid

Many of the operating costs of a greenhouse depend on its volume. For example, the energy used to heat a building depends on the volume of the building. The two large greenhouses at the Muttart Conservatory have square bases measuring 26 m on each side. The apex of each greenhouse is 24 m high. What is the volume of each greenhouse, to the nearest cubic metre?

Solution

Since the base is a square, $l = w = 26$, and $h = 24$.

Substitute into the formula $V = \frac{1}{3}lwh$.

$V = \frac{1}{3}lwh$

$V = \frac{1}{3}(26)(26)(24)$

$V = 5408$

The volume of each of the large greenhouses is 5408 m^3.

Your Turn

a) The Muttart Conservatory also has two smaller greenhouses. The base of each greenhouse is a square with side length 19.5 m, and the height of each greenhouse is 18 m. What is the volume of each of the smaller greenhouses?

b) If the smaller greenhouse had been designed as a right rectangular prism with the same size base, what would its height have to be in order for the greenhouse to have the same volume?

Did You Know?

Archimedes is attributed with the discovery of the formula for finding the volume of a sphere. He is considered by most historians of mathematics as one of the greatest mathematicians of all time. Before his death, he requested that his tomb include a monument featuring a stone sphere and cylinder.

Example 3 Calculate an Unknown Dimension When Given a Volume

The volume of an exercise ball is approximately 4188.8 cm^3. What is the diameter of this ball, in centimetres?

Solution

Substitute into the formula for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$
$$4188.8 \approx \frac{4}{3}\pi r^3$$
$$3(4188.8) \approx 3\left(\frac{4}{3}\pi r^3\right)$$
$$12\,566.4 \approx 4\pi r^3$$
$$\frac{12\,566.4}{4\pi} \approx \frac{4\pi r^3}{4\pi}$$
$$1000 \approx r^3$$
$$\sqrt[3]{1000} \approx \sqrt[3]{r^3}$$
$$10 \approx r$$

The symbol $\sqrt[3]{\ }$ indicates the cube root of a number. $\sqrt[3]{1000} = 10$ because $10 \times 10 \times 10 = 1000$.

The radius is approximately 10 cm.
Therefore, the diameter of the ball is approximately 20 cm.

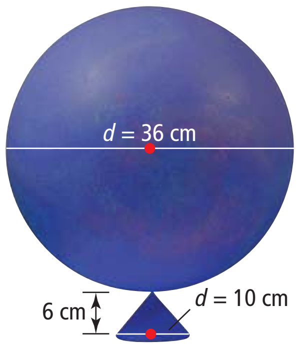
Your Turn

a) Find the cube root of 343.

b) Find the diameter, correct to the nearest millimetre, of a sphere with volume 288π cm^3.

Example 4 Finding the Volume of Composite Figures

Esther is creating a clay sculpture that includes a sphere attached to a right cone. What volume of clay, in cubic centimetres, does she need to make the sculpture?



Solution

Volume of Sphere

$$V = \frac{4}{3}\pi r^3$$
$$V = \frac{4}{3}\pi(18)^3$$
$$V = 7776\pi$$

Volume of Cone

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi(5)^2(6)$$
$$V = 50\pi$$

The diameter is twice the radius. The radius of the sphere is $\frac{36}{2}$ or 18 cm. The radius of the cone is $\frac{10}{2}$ or 5 cm.

Volume of Entire Sculpture

total volume = volume of sphere + volume of cone

$= 7776\pi + 50\pi$

$= 7826\pi$

The volume of the sculpture is 7826π cm^3, which is approximately 24 586 cm^3.

Your Turn

The Dominion Astrophysical Observatory near Victoria, BC, has a cylindrical base with a diameter of 20.1 m and a height of 9.8 m. The dome is half a sphere with the same diameter as the cylindrical base. What is the volume of the observatory?

Key Ideas

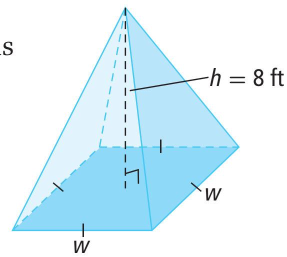
- The volume of a right cone is found by calculating one third of the volume of its related right cylinder.

  $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$
- The volume of a right pyramid is found by calculating one third of the volume of its related right prism.

  $V_{\text{pyramid}} = \frac{1}{3}lwh$
- The volume of a sphere is found by using the formula

  $V_{\text{sphere}} = \frac{4}{3}\pi r^3$
- If you know the volume of an object, you can calculate an unknown dimension.

  The volume of the right pyramid with square base is 384 ft^3. Find the dimensions of the base.



$$V = \frac{1}{3}Bh$$
$$384 = \frac{1}{3}w^2(8)$$
$$3(384) = 3\left(\frac{1}{3}w^2(8)\right)$$
$$1152 = w^2(8)$$
$$\frac{1152}{8} = w^2$$
$$144 = w^2$$
$$12 = w$$

The dimensions of the base are 12 ft × 12 ft.

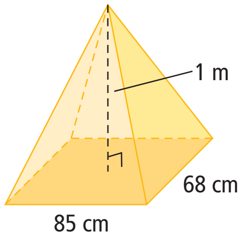
Check Your Understanding

Practise

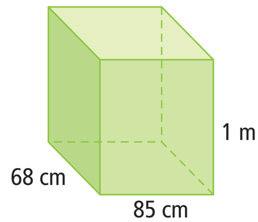
Where necessary, express your answers to the nearest tenth of a unit.

1. Calculate the volume of each of the following.

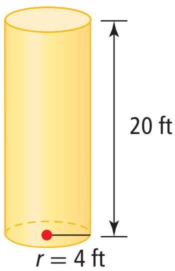
a)



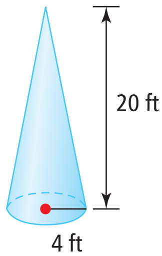
b)



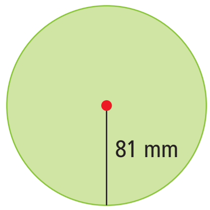
c)



d)

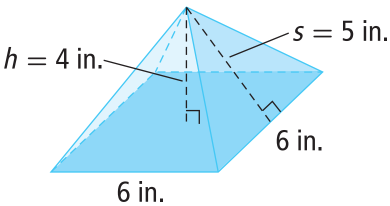


e)

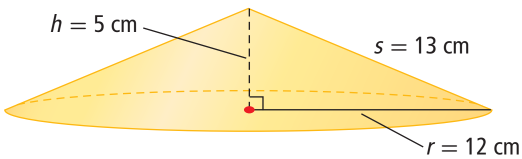


2. Calculate the volume of each solid.

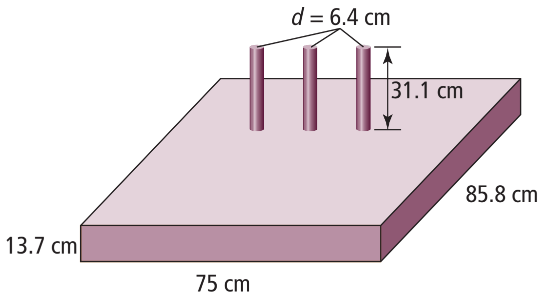
a)



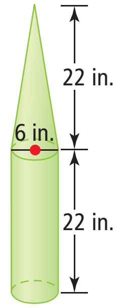
b)



3. Calculate the volume of the following composite object, in cubic metres.



4. Erin and Janine were asked to find the volume of this composite figure. Their work is shown.



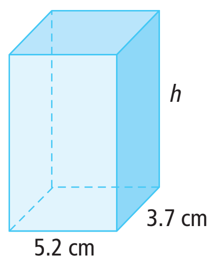
Erin	*Janine*
$V = \frac{1}{3}\pi r^2 h + \pi r^2 h$	$V = \frac{1}{3}(\pi r^2 h + \pi r^2 h)$
$V = \frac{1}{3}\pi(3^2)(22) + \pi(3^2)(22)$	$V = \frac{1}{3}(\pi(3^2)(22) + \pi(3^2)(22))$
$V = \frac{1}{3}198\pi + 198\pi$	$V = \frac{1}{3}(198\pi + 198\pi)$
$V = 66\pi + 198\pi$	$V = \frac{1}{3}(396\pi)$
$V = 264\pi$	$V = 132\pi$
$V \approx 829.38$	$V \approx 414.69$
The volume is 829.38 in.^{3}	The volume is 414.69 in.^{3}

a) Discuss with a partner which student correctly calculated the volume. Justify your answer.

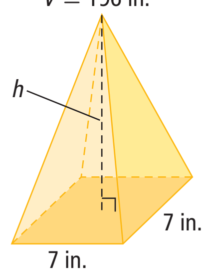
b) Identify which method you would use to calculate the volume of the composite figure.

5. Calculate the missing dimension for each of the following.

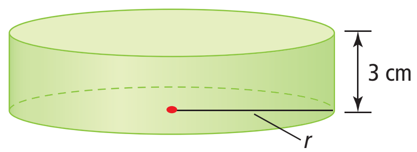
a) $V = 161.6$ cm^3



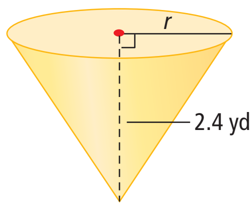
b) $V = 196$ in.^{3}



c) $V = 339.3$ cm^3



d) $V = 8.1$ yd^3



Apply

6. The Alliance Pipeline begins in northeastern British Columbia and is 3017 km long. It is the longest pipeline in North America. It carries crude oil, as well as natural gas, from exploration sites to markets. One of the longest sections of the pipeline is 1221.73 km long, with a diameter of 914 mm. The pipeline does not always run in a straight line. If it was straightened, what is the maximum volume of oil that can be contained in this section of pipeline? Express your answer to the nearest hundredth of a cubic metre.

Alliance Pipeline

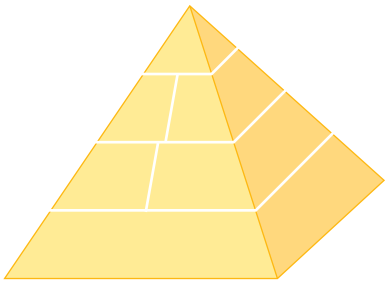
7. A right rectangular prism measures 9 in. × 4 in. × 6 in. What would be the dimensions of a cube with the same volume?

Did You Know?

Jade is the official gemstone of British Columbia. It is found in large deposits in the Lillooet and Cassiar regions. Its translucent emerald-green appearance makes it a popular choice for jewellery and other tourism memorabilia.

8. Kendra wants to purchase a bead necklace made of jade. The cost of the necklace depends upon the amount of jade in the necklace. Each bead is 7 mm in diameter and there are 100 beads in the necklace. What is the amount of jade in the necklace, in cubic centimetres?

9. In her art class, Andrea wants to make a locking puzzle in the shape of a pyramid to have on her desk. The base will measure 6 cm by 8 cm, and the height will be 10 cm. Calculate the amount of wood needed to create the pyramid.



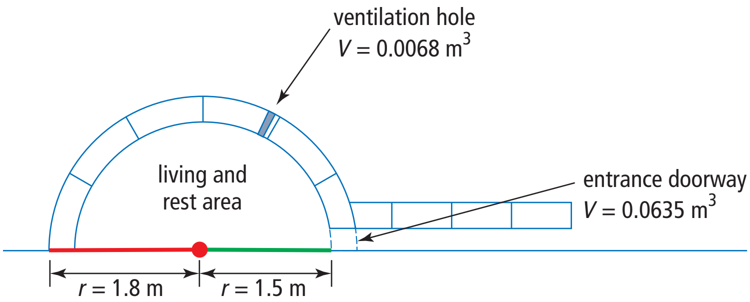
10. For each of the following objects, the volume is given. Sketch the object and calculate the unknown dimension.

a) A right cylinder has a volume of 500 cm^3 and a height of 16 cm. Calculate its radius.

b) A right cone with a volume of 20 cm^3 has a diameter of 5 cm. Calculate its height.

c) A sphere has a volume of 48 cm^3. Calculate the radius.

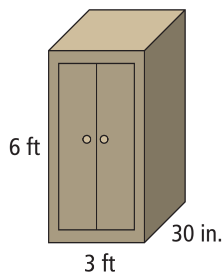
11. Traditionally, the Inuit of northwestern Canada have built domed-shaped family homes called igloos. The volume of snow in an igloo varies depending on the size. Every igloo has a ventilation opening as well as an entrance.



Calculate the volume of snow used to construct the main portion of igloo in the picture, not including the entrance tunnel. Express your answer to the nearest tenth of a cubic metre.

12. The roof of a house is shaped like a right pyramid with a square base. The base of the pyramid measures 32 ft on each side, and the roof must enclose a volume of at least 4096 ft^3 of air. Calculate the minimum height for the apex of the roof.

13. Grady has designed a wooden storage cabinet for his CDs and DVDs. The cabinet has a wooden door that can be closed. The cabinet is a right rectangular prism, as shown.

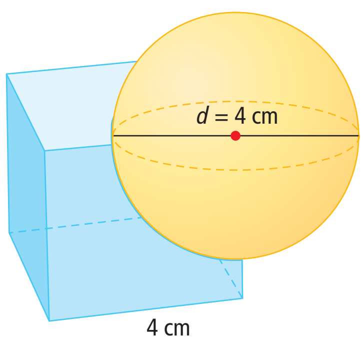


a) Calculate the volume of the cabinet.

b) Grady is concerned that the cabinet is too big for his room. He considers keeping the length and width the same, but reducing the height by one quarter. Estimate the volume of the modified cabinet.

c) Calculate the volume of the cabinet in part b). Was your estimate accurate? Explain how changing one dimension of a right rectangular prism by a factor of *k* will change the volume of the prism.

14. International Games are being held in your community. Your promotions committee designed a souvenir consisting of a cube with an inserted sphere representing the Earth. Both pieces are constructed from solid crystal. The sphere replaces one quarter of the volume of the cube. Calculate the volume of the souvenir piece.



Did You Know?

The word *igloo*, meaning a house of snow. The igloo shape is semi-spherical because it creates the greatest amount of living space with the least amount of snow.

Did You Know?

The first cellular phones to be created were very large and bulky. The first cell phone came to the market in 1984 from Motorola and weighed 2 lb. It was a DynaTac 8000X and it sold for $3995.

Web Link

To view a video that describes the cell phone development by Motorola, go to www.mhrmath10.ca and follow the links.

15. **Unit Project** The first cell phones were much larger than present-day cell phones. A typical cell phone now has a volume between 4 in.^{3} and 6 in.^{3}. Using the information shown in the photo, estimate the volume of the first commercial portable cell phone released in 1984 by Motorola.

16. **Unit Project** An MP3 player with a memory of 80 GB has a storage capacity of 20 000 average-length songs. A vinyl LP record is 0.11 in. thick and on average can hold 12 songs. If the dimensions of the MP3 player are 4.14 cm wide, 9.15 cm high, and 0.85 cm thick, and the record has a radius 6 in., how many songs per cubic centimetre are there on each storage medium? Express your answers to the nearest hundredth.

Extend

For each of the following, express your answers to the nearest tenth of a unit where necessary.

17. Through your work with volume, you discovered a relationship between the right cone and the right cylinder. Extend your work to describe a relationship between a right cone and a sphere.

18. Kelly works at an ice cream shop. Customers can choose to have a cone that is lined with chocolate. The store buys the chocolate-lined cones, but the owners are wondering if they could save money by lining their own cones. To decide, they ask Kelly to calculate the amount of chocolate used to coat the inside of each cone with a layer of chocolate 1 mm thick. Each cone has an inside diameter of 5.5 cm and a slant height of 13 cm. Calculate the volume of chocolate used to line the inside of each cone.

19. Use spreadsheet software to help investigate how changing the radius of a sphere changes its volume. Create a spreadsheet like the one shown below.

	A	B	C	D
1	Investigating Changes in Dimensions of a Sphere			
2	Stretch Factor	Radius	Volume	Ratio of New Volume to Original Volume
3	1	3	113.1	1
4	2	6		
5	3			
6	4			
7	5			

a) Use spreadsheet formulas to complete the spreadsheet. Depending on your software, you may need to type "PI" or "PI()" for π. See your spreadsheet's help feature if you need assistance.

b) Compare the stretch factor for the radius to the ratio of the new volume to the original volume. What pattern do you notice?

c) Use your pattern to predict the volume of the sphere if you multiply the radius by 6. Extend your spreadsheet to check your answer.

d) In your own words, express the relationship between a change in the radius of a sphere and its volume.

20. Choose another solid that you have studied in this section. Create a table to investigate how the volume of the solid is affected by changing one of the dimensions of the solid.

Create Connections

21. **Unit Project** Work individually or in a small group. Choose a 3-D object related to your Unit 1 project.

a) Estimate its volume in both SI and imperial units. Are your estimates reasonable? Explain.

b) Calculate the volume. Are the units in your answer appropriate for the object?

c) In which measurement system was your estimate more accurate? Why do you think this happened?