Solving Systems of Linear Equations by Elimination

Focus on ...

- writing equivalent equations to eliminate a variable
- solving systems of linear equations algebraically using elimination

Did You Know?

Fair trade products are goods from developing countries. They are priced to ensure that the producers receive fair payment.

Environmental clubs promote many activities and products that are environmentally friendly. They promote chemical-free foods such as organic coffee. They also promote products that minimize waste. These include reusable shopping bags, and food and drink containers. What other similar products can you think of?

For their fundraiser, an environmental club is selling reusable shopping bags and organic fair-trade coffee. Two recent sales were:
- three bags and two packets of coffee for $17
- one bag and one packet of coffee for $7

How could you determine the unit price of one bag and one packet of coffee?

Investigate Solving Systems of Equations by Elimination

In the following balance diagrams, each block is identical in mass. Each cone is identical in mass.

Diagram 1

A   B

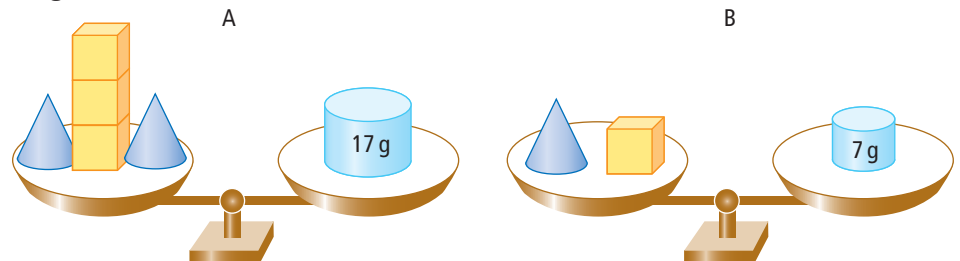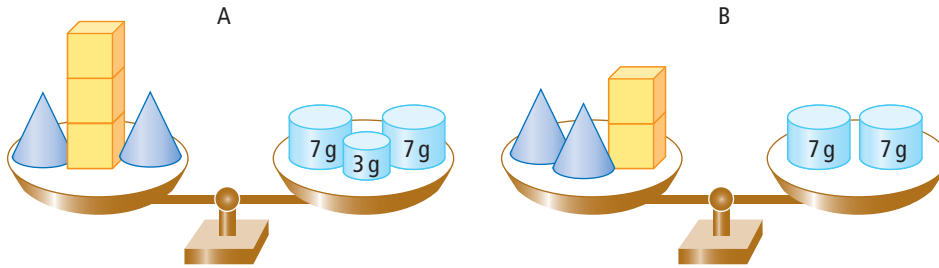17 g   7 g

Diagram 2

1. Explain why scale B in Diagram 2 is balanced.

2. Draw a diagram of a scale balance to show how to determine the mass of one block.

3. Explain how you can determine the mass of the cone given the mass of the block.

4. Write equations for each scale in Diagram 1.

5. Use algebra to show how you would change scale B in Diagram 1 in order to write the equation for scale B in Diagram 2.

6. Use algebra to show how you can write the equation for the scale in your diagram from the two equations in Diagram 2.

7. Suppose the mass of a block represents the cost of one bag, the mass of a cone represents the cost of one packet of coffee, and each gram represents one dollar. How can you determine the cost of one bag and the cost of one coffee packet using algebra for the scenario on page 480?

8. **Reflect and Respond** Use diagrams to explain how to determine the mass of a single pyramid and the mass of a single cylinder for the following scenario.
   - Four pyramids and three cylinders have a mass of 23 g.
   - Two pyramids and five cylinders have a mass of 29 g.

   pyramid   cylinder

9. Determine algebraically the mass of one pyramid, p, and the mass of one cylinder, c.

10. Could you use the substitution method to solve this Investigate? Explain.
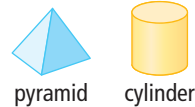
Link the Ideas

You can solve a system of linear equations using the **elimination method**. To do this, a variable in both equations must have the same or opposite coefficient. It is often necessary to multiply one or both equations by a constant.

For example, solve the following linear system:

$$6a + 5b = 24$$
$$4a + 3b = 12$$

In order to eliminate variable a, you need to multiply the first equation by 2. Multiply the second equation by 3. Now, both equations will contain the term $12a$.

$$2(6a + 5b) = 2(24)$$
$$3(4a + 3b) = 3(12)$$

Why should you choose a constant with the smallest possible value?

Example 1   Solve a System of Linear Equations by Elimination

Connor downloaded two orders of games and songs. The first order consisted of five games and four songs for $26. The second order consisted of three games and two songs for $15. All games cost the same amount, and all songs cost the same amount. Write a system of linear equations. Then, determine the cost of one song and the cost of one game.

Solution

Let S represent the cost of one downloaded song, in dollars.
Let G represent the cost of one downloaded game, in dollars.

Write two linear equations. Write an equation to represent the first order.

$$5G + 4S = 26 \quad ①$$

How does ① represent the cost of five games and four songs?

Write an equation to represent the second order.

$$3G + 2S = 15 \quad ②$$

Determine which variable to eliminate. One strategy is to examine each variable in both equations. Look for a least common multiple for the coefficients of the G terms or the S terms.

$$5G + ④S = 26$$
$$3G + ②S = 15$$

The coefficients of the term $4S$ in ① and the term $2S$ in ② have a least common multiple of 4.

Multiply ② by −2 so that there is an opposite S term to $-4S$ in ①. Then, you can add the equations to eliminate the S term.

You can also multiply by 2. Then, you can subtract the equations.

$$-2(3G + 2S) = -2(15)$$
$$-6G - 4S = -30 \qquad ③$$

Add ③ and ① to eliminate the S terms.

$$
\begin{array}{r}
-6G - 4S = -30 \\
+ (5G + 4S = 26) \\
\hline
-G = -4 \\
G = 4
\end{array}
$$

Solve for *G*.

What does the value 4 represent?

Solve for the remaining variable, S, by substituting 4 for G in ① or ②.

$$5(4) + 4S = 26$$
$$20 + 4S = 26$$
$$4S = 6$$
$$S = 1.5$$

What does the value 1.5 represent?

Check:
Substitute into ① and ②.

Left Side	Right Side
$5G + 4S$	26
$= 5(4) + 4(1.5)$	
$= 20 + 6$	
$= 26$	

Left Side = Right Side

Left Side	Right Side
$3G + 2S$	15
$= 3(4) + 2(1.5)$	
$= 12 + 3$	
$= 15$	

Left Side = Right Side

The cost of one game is $4.00, and the cost of one song is $1.50.

Your Turn

A group of people bought tickets for a University of Alberta basketball playoff game. Two student tickets and six adult tickets cost $102. Eight student tickets and three adult tickets cost $114. What was the price for a single adult ticket? What was the price for a single student ticket?

Example 2   Solve a System of Linear Equations Using a Table and Elimination

A crop farmer has contracted with the Pacific Carbon Trust (PCT) to convert some of her cropland into woodland. This will create a carbon sink that is used to offset the production of carbon resulting from her farm activities. The farmer has 500 ha of cropland. She earns approximately $220/ha from the crops. The PCT will pay her $60 for every hectare of cropland that she converts. She would like a minimum revenue of $90 800 that year. Using the elimination method, determine the number of hectares that she needs to convert to woodland. How many hectares of cropland would be left?

Solution

Let c represent the number of hectares of cropland.
Let w represent the number of hectares of woodland.
Organize the information in a table.

Type of Land	Revenue Generated Per Hectare ($)	Number of Hectares	Revenue Generated ($)
Cropland	220	c	$220c$
Woodland	60	w	$60w$
Total		500	90 800

Write an equation to show the total number of hectares.
$c + w = 500$ ①

Write an equation to determine the revenue created.
$220c + 60w = 90\ 800$ ②   **The farmer wants a minimum revenue of $90 800.**

Determine which variable to eliminate. One strategy is to examine each variable in both equations. Then, identify the coefficient, other than 1, that is closest to zero.

$c + w = 500$

$220c + \boxed{60}w = 90\ 800$ 　　　**The term 60w in ② has the coefficient closest to zero.**

Multiply ① by -60 so that there is an opposite w term.

$-60(c + w) = -60(500)$ 　　**What do you multiply ① by if you want to**

$-60c - 60w = -30\ 000$ 　③ 　**subtract the equations?**

Add ② and ③ to eliminate the w terms.

$$
\begin{array}{rl}
220c + 60w = & 90\ 800 \\
+ \ (-60c - 60w = & -30\ 000) \\
\hline
160c \quad\quad = & 60\ 800 \\
c \quad\quad = & 380
\end{array}
$$

What does the value 380 represent?

Solve for the remaining variable, w, by substitution.

$380 + w = 500$ 　　**Why is it more efficient to use ① instead of**

$\qquad w = 120$ 　　**② to solve for w?**

　　　　　　　　　　What does the value 120 represent?

Check:

Substitute into ① and ②.

Left Side 　　　　　　Right Side

$\quad c + w$ 　　　　　　　500

$= 380 + 120$

$= 500$

　　　Left Side = Right Side

Left Side 　　　　　　Right Side

$\quad 220c + 60w$ 　　　　90 800

$= 220(380) + 60(120)$

$= 83\ 600 + 7200$

$= 90\ 800$

　　　Left Side = Right Side

To generate a revenue of $90 800, the farmer could convert up to 120 ha to woodland. This would leave 380 ha for cropland.

Your Turn

During lunch, the cafeteria sold a total of 160 muffins and individual yogurts. The price of each muffin is $1.50. Each container of yogurt is $2.00. The cafeteria collected $273.50. Set up and solve a linear system in order to determine the number of muffins and the number of yogurts sold.

Example 3 Solve a System of Linear Equations in Different Forms by Elimination

The perimeter of a rectangular garden is 17.00 m. Triple the length is 2.46 m longer than five times the width. Sketch and label a diagram. Create a system of linear equations to determine the dimensions of the rectangle. Solve the system using elimination.

Solution

l

$P = 17.00 \text{ m}$ w
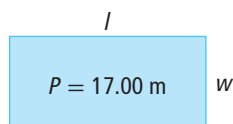
Let w represent the width of the rectangle, in metres.
Let l represent the length of the rectangle, in metres.

Write an equation to represent the perimeter.
$2w + 2l = 17.00$ ①

Write an equation to represent the difference in the dimensions.
$3l = 5w + 2.46$ ②

Rearrange ② so that it is in the form $ax + by = c$, similar to ①.
$$3l - 5w = 5w + 2.46 - 5w$$
$$-5w + 3l = 2.46 \quad ③$$

Decide whether you need to multiply one or both equations by a constant to eliminate a variable. Multiply ① by 3 and multiply ③ by -2 so that there are opposite l terms.
$$3(2w + 2l) = 3(17.00)$$
$$6w + 6l = 51.00 \quad ④$$

$$-2(-5w + 3l) = -2(2.46)$$
$$10w - 6l = -4.92 \quad ⑤$$

Add ④ and ⑤ to eliminate l.

$$10w - 6l = -4.92$$
$$+ (6w + 6l = 51.00)$$
$$16w \quad\quad = 46.08 \quad\quad \textbf{Solve for } \textit{w}.$$
$$w \quad\quad = 2.88$$

Solve for l by substituting into ②.

$3l = 5w + 2.46$
$3l = 5(2.88) + 2.46$
$3l = 14.40 + 2.46$
$3l = 16.86$
$l = 5.62$

Check:
Substitute into ① and ②.

Left Side	Right Side
$2w + 2l$	17.00
$= 2(2.88) + 2(5.62)$	
$= 5.76 + 11.24$	
$= 17.00$	

Left Side = Right Side

Left Side	Right Side
$3l$	$5w + 2.46$
$= 3(5.62)$	$= 5(2.88) + 2.46$
$= 16.86$	$= 14.40 + 2.46$
	$= 16.86$

Left Side = Right Side

The garden has a width of 2.88 m. Its length is 5.62 m.

Your Turn

A rectangular parking pad for a car has a perimeter of 12.2 m. The width is 0.7 m shorter than the length. Use a linear system to determine the dimensions of the pad.

Key Ideas

- A table can help you organize information in a problem. This can help you to determine the equations in a linear system.

- You can solve a linear system by elimination.

$$3x + 2y + 6 = 0$$
$$7y = 5x + 41$$

  - If necessary, rearrange the equations so that like variables appear in the same position in both equations. The most common form is $ax + by = c$.

$$3x + 2y + 6 = 0 \qquad\qquad 7y = 5x + 41$$
$$3x + 2y = -6 \quad ① \qquad -5x + 7y = 41 \quad ②$$

  - Determine which variable to eliminate. If necessary, multiply one or both equations by a constant to eliminate the variable by addition or subtraction.

    Multiply ① by 5 and ② by 3 so that the coefficients of the terms involving x add to zero.

$$5(3x + 2y) = 5(-6) \qquad 3(-5x + 7y) = 3(41)$$
$$15x + 10y = -30 \quad ③ \qquad -15x + 21y = 123 \quad ④$$

    Add ③ and ④ to eliminate x.

$$\begin{aligned} 15x + 10y &= -30 \\ + \underline{(-15x + 21y = 123)} \\ 31y &= 93 \end{aligned}$$

    You can also multiply by −3 or by −5 and then subtract.

  - Solve for the remaining variable.

$$31y = 93$$
$$y = 3$$

  - Solve for the second variable by substituting the value for the first variable into one of the original equations.

$$7(3) = 5x + 41$$
$$21 = 5x + 41$$
$$-20 = 5x$$
$$-4 = x$$

- Check your solution by substituting each value into both original equations.

Check Your Understanding

Practise

1. Solve using elimination.

a) $x + y = 10$
  $x - y = 4$

b) $x + 2y = 13$
  $x - y = 8$

c) $y - 2x = -4$
  $y + 3x = 16$

2. Rearrange the equations so that the variables are ordered in the same way for both equations.

a) $y - 3x = 11$
$x - y = -5$

b) $x + 7 = y$
$2x + y = -8$

c) $4 - 3y = x$
$x - y = 16$

3. Use the table to set up a linear system. Use the elimination method to determine the number of tickets sold to:

a) adults   **b)** students

Ticket Type	Price Per Ticket ($)	Number of Tickets Sold by Type	Revenue Collected by Theatre ($)
Students	10	s	$10s$
Adults	13	a	$13a$
Total		430	4804

4. Solve the following systems of linear equations by elimination. Check your answers.

a) $3x + 2y = 7$
$4x + 5y = 14$

b) $7x - 6y = 27$
$2x + 9y = -3$

c) $4y + 29 = 3x$
$8x + 7 = 3y$

5. Solve using the elimination method. Leave your answers in fraction form.

a) $3x + 2y = 10$
$2x - y = 4$

b) $\frac{x}{3} - y = \frac{3}{5}$
$x + 6y = 4$

c) $2 - \frac{y}{2} = \frac{x}{3}$
$\frac{2}{3}(2x - 3y) = 4$

6. Solve the following system. Explain the result.

$3x + 2y = 7$
$9x + 6y = 16$

Apply

Solve problems 7 to 14 using the elimination method. Check your answers.

7. A preschool playground has both bicycles and tricycles. There is a total of 30 seats and 70 wheels. How many bicycles are there? How many tricycles are there?

8. Students at Evergreen High School want to help the community with the Communities in Bloom project. They decide to sell flower bulbs to raise money. Nancy sells 10 bags of tulip bulbs and 12 bags of iris bulbs for $380. Shawn sells 6 bags of tulip bulbs and 8 bags of iris bulbs for $244. What is the cost of one bag of tulip bulbs? What is the cost of one bag of iris bulbs?

Did You Know?

Communities in Bloom is a national non-profit organization dedicated to the creation and maintenance of green spaces in urban settings. On September 20, 2008, Lethbridge, AB, hosted the 14th Communities in Bloom National Awards Ceremony.

9. At the snack bar, five toasted bagels and three cans of juice cost $12.50. Three toasted bagels and six cans of juice cost $12.75. What is the price for one bagel? What is the price for one juice?

10. A total of 430 dogs and people attended the Woof Walk fundraiser. Altogether, 1210 legs participated in the walk. How many dogs were there?

11. A ferry is carrying 600 vehicles, including trucks and passenger vehicles. The fees collected total $29 200. The charge per truck is $100. The charge per passenger vehicle is $45. How many trucks and how many passenger vehicles is the ferry carrying?

12. An avalanche rescue team travels 8.55 km along a snow-covered trail. For the first section, the trail is flat. The team averages a speed of 2.7 km/h. Then, the terrain becomes mountainous and their average speed is only 1.2 km/h. The one-way trip takes the team 4.0 h. Determine the distance that the team travels on each type of terrain.

13. Soo Jin had basketball practice after school. Then, she cycled home. Playing basketball, she expends energy at a rate of 25 kJ per minute. Cycling home, she burns energy at a rate of 21 kJ per minute. She spent a total of 90 min doing both forms of exercise. During this time, she expended a total of 2178 kJ of energy. How much time did she spend doing each activity?

14. **Unit Project** Sharon estimates that she saves 260 L of water per week by washing her car with a bucket and sponge. Her sister Bev washes her car with a hose, which uses more water. Sharon's washing machine uses 225 L of water per load. Bev has upgraded to a washing machine that uses only 95 L of water per load. Both sisters wash the same number of loads of laundry per week. Both wash their car once a week.

a) Develop a system of equations representing their water usage in one week.

b) When their water usage is the same, how many loads of laundry does each sister do in one week?

c) If each sister does eight loads of laundry per week, who uses more water weekly? Explain.

Extend

15. Simplify. Then, solve the following linear systems using elimination.

a) $3(x + 2) + 7y = 11$
$-5(3 - x) + 9y = -12$

b) $5x - 2(y + 4) = y - 3x$
$2(x + 8y) - 4y = 9x$

16. Brittany invested a total of $3000 in two different investments. The safer investment earned 3.5% interest by the end of the year. The riskier investment earned 5.2% interest by the end of the year. Her total interest earned was $126.25. How much did she invest in the safer investment?

17. Milk that has 3.25% milk fat (MF) is mixed with milk that has 1% MF. What volume of each is needed to obtain 60 L of milk that has 3% MF? Express your answers to the nearest tenth of a litre.

18. Determine the value of k so that $6x + 4y = 7$ and $kx + 8y = 7$ do not have a common solution.

19. The solution to the system $10x + 12y = -18$ and $5x + 4y = b$ is $(9, a)$. What can b and a be? Is there more than one possibility? Explain.

Create Connections

20. a) Choose a question from section 9.2 and solve it using the substitution method.

b) Explain why you selected the question you did.

c) Are there any questions in section 9.2 that do not suit being solved using the substitution method? Explain.

21. What do you need to consider when choosing whether to use the substitution method or the elimination method to solve a system of linear equations? Provide examples to clarify your explanation.