8.2 Modelling and Solving Linear Systems

Focus on ...

- translating word problems into systems of linear equations
- interpreting information from the graph of a linear system
- solving problems involving systems of linear equations

A big part of travelling involves choosing the mode of transportation. Different modes of transportation usually take different lengths of time, have different costs, and may have different effects on the environment. *The Canadian* is a passenger train that travels across Canada, between Vancouver and Toronto. How might taking *The Canadian* compare with other modes of travel? In what ways could you represent your comparison?

Materials

- map of Canada
- ruler
- grid paper or computer with graphing software

Investigate Creating a System of Linear Equations

1. Work with a partner. Select two cities in Canada that are in different provinces. Make sure each city has passenger rail service. Research the distance between the cities.

2. Suppose both of you need to travel from city A to city B. One person will drive and the other person will take a train. Assign the modes of travel. Then, determine the travel time for your particular mode. Assume the average speed of the car is 90 km/h and the average speed of the train is 70 km/h.

3. Assign a variable to represent each measurement.

a) the distance from city A

b) the length of time that the car has been travelling

4. a) Write an equation to model the travel of the car.

b) Suppose the train leaves 1 h before the car. Write an equation that represents the travel of the train.

5. Graph the system of linear equations you developed. What is the solution to your linear system? What does the solution represent? Discuss your answers with your partner.

6. Suppose the car leaves at 7:30 a.m. and the train leaves at 9:00 a.m.
   a) Write a system of linear equations representing the travel from city A to city B. Compare your linear system with your partner's system.
   b) Solve your linear system by graphing. What does the solution represent?

7. **Reflect and Respond**
   a) Which system of equations did you find easier to write? Explain why. Share your strategies with a classmate.
   b) How are the two equations in each linear system similar? How do they differ? Explain.
   c) How are the information given in a problem, the equations, and the graph connected? Provide examples to support your explanation.

8. Why can this situation be represented with a system of linear equations? Explain.

Link the Ideas

The ability to translate words or phrases into the language of mathematics is an important skill for solving problems in context. There are a limited number of mathematical operations, but there are many phrases that can be used to describe the operations.

For example, the following situations can all be represented by the expression $7x + 3$.

- 7 km/h for a length of time and then three more kilometres
- $7 per person plus $3
- 7 times as many years increased by 3

When it comes to subtraction, the translation requires greater consideration.

$x - 3 \neq 3 - x$ when $x \neq 3$

For example, 3 less than a number means $x - 3$ whereas 3 decreased by a number means $3 - x$.

Situations involving quantities that change at constant rates can be represented algebraically with a system of linear equations.

Example 1 Model a Linear System Algebraically and Graphically

People can rent ski and snowboard equipment from two places at Winterland Resort.
Option A charges a one-time \$30 fee and then \$8 per hour.
Option B charges \$14 per hour.

a) Create a system of linear equations to model the rental charges.
b) Solve the linear system graphically. What does the solution represent?

Solution

a) Both rental options involve a constant rate per hour, so they represent linear relations.
Identify the unknown values and assign variables.
Let C represent the cost, in dollars.
Let t represent the length of time, in hours, of the rental.

Write an equation to model the cost for each rental option.

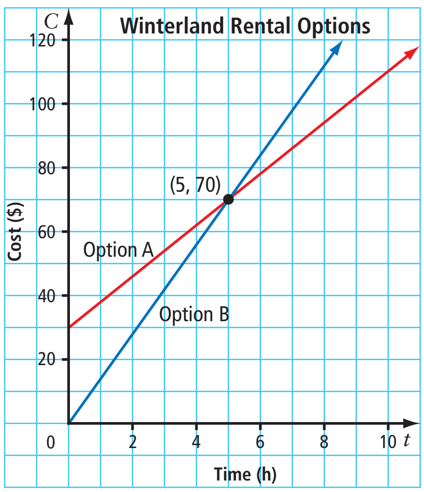
Option A: The cost is \$30 plus \$8 per hour.
$C = 30 + 8t$ — How is this equation related to the slope-intercept form?

Option B: The initial value is \$0 and the rate per hour is \$14.
The cost is \$14 per hour.
$C = 14t$
The equations $C = 30 + 8t$ and $C = 14t$ form a linear system.

b) To solve the linear system $C = 30 + 8t$ and $C = 14t$, graph the equations together and identify the point of intersection.

Method 1: Use Paper and Pencil

Graph the two equations.



What is the length of rental when both options have the same charge? What is that amount? How would you decide which option is better for you?

From the graph, the point of intersection is (5, 70). This is the solution to the linear system. It represents the length of rental when both options have the same charge.

The solution (5, 70) can be verified by substitution.

Option A: $C = 30 + 8t$

Left Side	Right Side
C	$30 + 8t$
$= 70$	$= 30 + 8(5)$
	$= 30 + 40$
	$= 70$

Left Side = Right Side

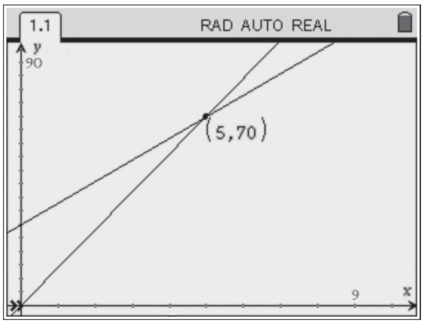
Option B: $C = 14t$

Left Side	Right Side
C	$14t$
$= 70$	$= 14(5)$
	$= 70$

Left Side = Right Side

Method 2: Use Technology

Graph the equations $C = 30 + 8t$ and $C = 14t$. You will likely need to use the equations $y = 30 + 8x$ and $y = 14x$.



How will you determine what range of values to plot?

Use the intersect feature to find the point of intersection, (5, 70). For a 5-h rental, both options cost $70.

The solution can be verified by substituting using technology.

$30+8\cdot5$	70
$14\cdot5$	70

Your Turn

During a performance by a theatre company, the main act was on stage for 3 min less than twice the time of the opening act. Together, the two acts performed for 132 min.

a) Write a system of linear equations to represent the length of time each act performed.

b) What is the solution to this linear system? What does the solution represent?

Did You Know?

La Troupe du Jour is the only professional francophone theatre company in Saskatchewan. It was founded in 1985 and develops French-language theatre for the community.

Example 2 Interpret Information From the Graph of a Linear System

Two hopper-bottom grain bins are being emptied starting at the same time.

- The larger bin holds 45 m^3 of grain. It is emptied at a rate of 1 m^3 per minute.
- The smaller bin stores 30 m^3 of grain. This bin is emptied at a rate of 0.5 m^3 per minute.

a) Model the volume of grain remaining as a function of time using a system of linear equations.

b) Represent the linear system graphically. Describe how the information shown in the graph relates to the grain bins.

Solution

a) Define the variables.

Let V represent the volume of grain remaining in each bin, in cubic metres.

Let t represent time, in minutes.

Organize the information using a table.

Bin	Starting Volume (m^3)	Volume of Grain Removed (m^3)	Volume of Grain Remaining in Bin, V (m^3)
Larger	45	t	$45 - t$
Smaller	30	$0.5t$	$30 - 0.5t$

The larger bin starts with 45 m^3. It is being emptied at a rate of 1 m^3/min.

$V = 45 - t$

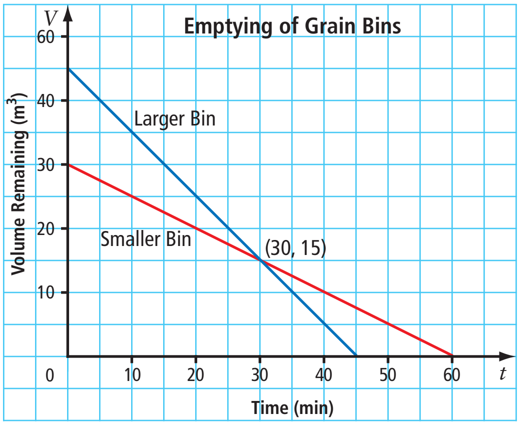
Why is the rate of change a negative value?

The smaller bin starts with 30 m^3. It is being emptied at a rate of 0.5 m^3/min.

$V = 30 - 0.5t$

A system of linear equations that models this situation is $V = 45 - t$ and $V = 30 - 0.5t$.

b) On the same grid, graph the system of linear equations, $V = 45 - t$ (larger bin) and $V = 30 - 0.5t$ (smaller bin).



The graph shows that the amount of grain remaining in each bin decreases over time. Both lines stop at the horizontal axis because the volume of grain left inside is 0 m^3.

The two lines intersect at (30, 15). This is the only point when the two bins contain the same amount of grain at the same time. At exactly 30 min, each bin has 15 m^3 of grain remaining.

Before 30 min, the line representing the larger bin is *above* the line for the smaller bin. This means that before 30 min, the larger bin has more grain inside. After 30 min, the line for the larger bin is *below* the line for the smaller bin. So, after 30 min, the smaller bin has more grain inside.

Your Turn

Two pools start draining at the same time. The larger pool contains 54 675 L of water and drains at a rate of 25 L/min. The smaller pool contains 35 400 L of water and drains at a rate of 10 L/min.

a) Model the draining of the pools algebraically using a system of linear equations.

b) Represent the linear system graphically. Describe how the information shown in the graph relates to the pools.

Example 3 Model and Solve a Problem Involving a Linear System

A movie theatre charges \$11 for an adult ticket and \$8 for children's or seniors' tickets. Suppose 240 people attended the early movie and ticket sales totalled \$2370.

a) The box office manager wants to know how many adults attended the early movie. What system of linear equations could help the manager determine the answer?

b) How many adults attended the early movie?

Solution

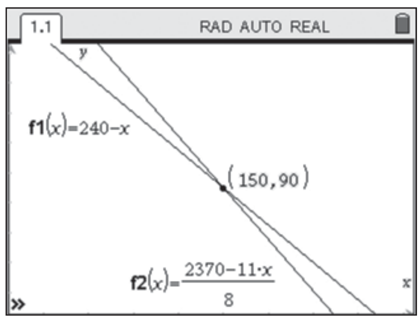
a) Define the variables.
Let a represent the number of adult tickets sold.
Let c represent the number of children's or seniors' tickets sold.

Write an equation to model the number of people at the early movie.
$a + c = 240$

Write an equation to model the ticket sales.
$11a + 8c = 2370$

The manager could use the linear system $a + c = 240$ and $11a + 8c = 2370$ to help determine the number of adults at the movie.

b) Graph both equations and identify the point of intersection.



The coordinates of the point of intersection are (150, 90). This is the only point that satisfies both equations. So, (150, 90) is the solution to the system of linear equations. There were 150 adults and 90 children and seniors at the early movie.

Your Turn

Jamie is travelling with her family from Castlegar, BC, to Pincher Creek, AB. Her dad and cousin do all of the driving. The 440-km trip takes 5.25 h, excluding stops. Jamie's father drives at an average speed of 90 km/h. Her cousin drives at 80 km/h. What system of linear equations could help Jamie determine the length of time each person drove? How many hours does each person drive?

Key Ideas

- When modelling word problems, assign variables that are meaningful to the context of the problem.
- To assist in visualizing or organizing a word problem, you can use a diagram and/or a table of values.
- If a situation involves quantities that change at constant rates, you can represent it using a system of linear equations.

  Two tanks are being filled at constant rates:
  One tank contains 100 L and is filling at 20 L/min.
  The other tank is empty and filling at 25 L/min.
  The situation can be represented by the system
  $V = 100 + 20t$ and $V = 25t$.
  In the equations, V represents volume, in litres, and
  t represents time, in minutes.

- If you know the initial values and rates, you can write the equations directly in slope-intercept form because the initial value is the y-intercept and the rate is the slope. Otherwise, you can determine the rate of change using start and end values.

  An electronics store charges a fee of \$36 plus \$6 per hour to fix a repair.
  The initial value is \$36, so $b = 36$.
  The rate is \$6 per hour, so $m = 6$.
  The equation is $C = 6t + 36$, where C represents cost, in dollars, and t represents time, in hours.

Check Your Understanding

Practise

Did You Know?

In 2009, Calgary began the Blue Cart recycling program. In this program, waste is sorted at a local plant owned by Metro Waste Paper Recovery. The plant is the most automated waste recovery facility in North America. It uses high-tech equipment and sorts 40 t of material every hour. In one year, the plant sorts and processes about 120 000 t of waste.

1. Model each situation using a system of linear equations.

a) One music download option costs \$0.99 per song. Another option costs \$11 plus \$0.79 per song.

b) A helicopter is 800 m above ground and descending at 55 m/min. An airplane is taking off and rising 80 m/min.

c) A recycling plant sorts material at a rate of 20 t per hour. It has sorted 100 t of material so far today. A new plant just opened. It sorts material at a rate of 40 t per hour.

2. Write a system of linear equations to represent each situation.

a) Jamal is three times as old as Maria. In seven years, he will be twice as old as she will be.

b) One day, the temperature in one city drops at a constant rate from 2 °C to −6 °C in 4 h. Meanwhile in another city, the temperature rises at a constant rate from −8 °C to 4 °C in 3 h.

3. Molly has a total of 32 points in her hockey league. One point is earned for an assist or a goal. If she has three times as many assists as goals, write a system of linear equations to represent how many goals and assists she has.

4. A collection of 50 coins contains only dimes and quarters. The value of the collection is \$6.80.

a) Use the table to write a system of linear equations relating the number of dimes to the number of quarters.

Type of Coin	Value of One Coin (\$)	Number of Coins	Value of Coins (\$)
Dime	0.10	d	$0.10d$
Quarter	0.25	q	$0.25q$

b) Rewrite the table, expressing the value of each type of coin in cents. Then, write a system of equations to model this relationship.

Apply

5. Two full water tanks drain at the same time. One tank holds 800 L of water. It drains at a rate of 30 L/min. The other tank holds 300 L of water. It drains at a rate of 12 L/min.

a) Express the draining of the water tanks algebraically using a system of linear equations.

b) Graph the linear system.

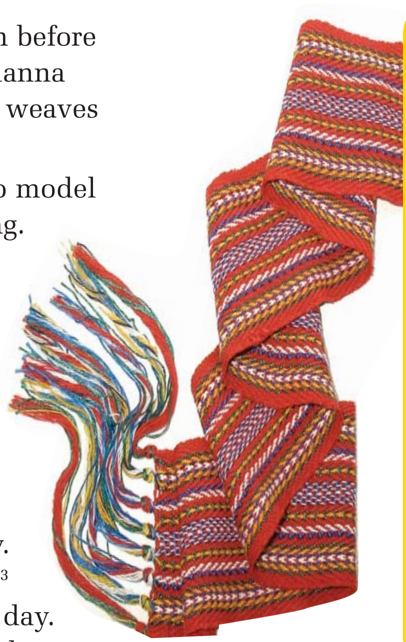
c) Explain how the information shown in the graph relates to the water tanks. What does the point of intersection represent?

6. Kianna weaves 45 cm of her Métis sash before Naomi starts weaving her own sash. Kianna weaves about 15 cm each hour. Naomi weaves about 25 cm every hour.

a) Write a system of linear equations to model the progress of the two girls' weaving.

b) Create a graph of the system of linear equations.

c) Describe how the information shown in the graph relates to the Métis sashes that the girls are weaving.



Did You Know?

Métis sashes are traditionally made with Aboriginal finger-weaving techniques. The weavers use European materials. In the past, the sash had many uses, including as a rope, washcloth, dog harness, baby carrier, and belt. Today, the sash is a symbol of pride. It is common for Prairie Métis organizations to bestow the Order of the Sash. There is no higher honour in the Métis community than receiving a sash as a gift.

7. One oil well has produced 2100 m^3. It produces oil at a rate of 7 m^3 per day. Another oil well has produced 1500 m^3 and produces oil at a rate of 15 m^3 per day. Suppose the two wells continue to produce oil at their current rates. When will both oil wells have produced the same amount? How much oil will they have produced at that time?

8. Megan considers two different car rental options. One option costs \$19 per day plus \$0.12 per kilometre. Another option costs \$42.50 per day for unlimited kilometres.

Compare the two options for a one-day rental. Which option should Megan choose? Why?

Web Link

To learn more about low-flow shower heads and other water-saving fixtures, go to www.mhrmath10.ca and follow the links.

Did You Know?

All Arctic communities except the very largest are on trucked water systems. So, most showers in the North use low-flow shower heads.

Did You Know?

The Test of Metal is a mountain bike race that takes place in Squamish, BC, every summer. The 67-km course includes many long, steep climbs and lots of technical off-road riding. More than 800 competitors of all ability levels compete. Finish times range from under 3 h to over 6 h.

9. **Unit Project** The Benoit family is deciding whether to replace their conventional shower head with a low-flow model. Their current shower head uses 170 L of water per 10-min shower. A typical low-flow shower head costs $25 to purchase. It uses 85 L per 10-min shower. Heating the water with electricity costs approximately $0.002 per litre.

a) If n represents the number of 10-min showers, write an expression for the cost of n showers using their current shower head.

b) Write a system of linear equations to represent the cost of showering using each type of shower head.

c) Graph the system of linear equations.

d) What is the solution to your linear system? What does it represent?

e) How would your solution change if each shower was reduced to 8 min? Justify your answer with a graphical analysis.

10. Amaruk and Mary are both travelling by boat between Igloolik and Hall Beach, NU. Amaruk is 35 km away from Hall Beach. He is travelling south to Hall Beach at a speed of 40 km/h. Mary is 15 km away from Hall Beach. She is travelling north at a speed of 25 km/h. When they pass each other, how far will they be from Hall Beach?

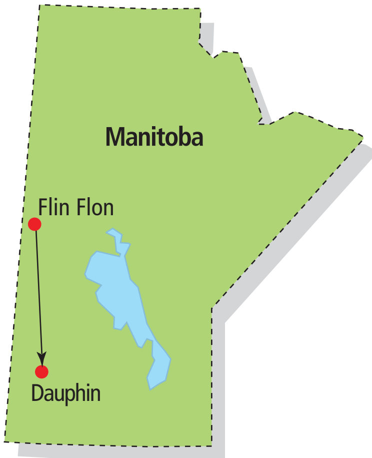
11. Caleb and Mitch compete in the Test of Metal mountain bike race. For a 2-km-long section, Caleb is 400 m up the hill and rides at a rate of 7.5 km/h. Mitch is at the base of the hill and rides at 10 km/h. Use a linear system to model this section of Caleb's and Mitch's race. Will Mitch catch Caleb before they reach the top of the hill? Explain why.

12. **Unit Project** A nearby wetland is estimated to have 100 ducks and 300 fish. A source of pollution seems to have contaminated the water. A local environmental group realizes that the number of ducks is decreasing at an average rate of 5 per year. The number of fish is decreasing at an average rate of 20 per year. Suppose the situation is considered critical if the number of fish equals the number of ducks in the area.

a) Write a system of linear equations to represent the numbers of fish and ducks in the wetland. Create a graph of your system.

b) Will the decreasing rates of fish and ducks become critical? If so, when? Justify your answer.

13. While driving from Flin Flon to Dauphin, MB, Kevin and his family had a flat tire. Before the flat tire, Kevin's parents drove at an average speed of 90 km/h. Once the flat tire was replaced with the spare tire, they travelled at an average speed of 75 km/h for the remainder of the trip. The total distance between the two cities is 538 km. The total driving time was 6 h. Write a linear system to model the family's travel. How far did they travel before the flat tire occurred? Include a labelled diagram.



14. Chris paints from one end of a 120-ft-long fence. He paints at a rate of 9 ft/h. Robert paints from the other end at a rate of 12 ft/h. Use a system of linear equations to determine when and where they will finish painting the fence.

15. Trevor is doing a project on tree growth rates. He measures the heights of two trees in early spring and again 20 days later. The younger tree grows from 120 cm to 130 cm tall. The older tree grows from 140 cm to 144 cm tall. Assuming each tree grows at a constant rate, when will the trees be the same height? What will this height be?

16. For part of the first year of a dog's life, its growth can be approximated using a linear function. Emilie has two puppies: a Border collie and a younger Saint Bernard. During a 4-week period, the Border collie grows from 13.4 kg to 17 kg, while the Saint Bernard grows from 6 kg to 12.4 kg. Suppose each dog grows at a constant rate. Will the dogs ever have the same mass? If so, approximately what is the mass?

17. A parachutist descends from a height of 500 m to 300 m above ground in 50 s. During the same time, a balloonist rises from 200 m to 450 m.

a) Write a system of linear equations to model their heights.

b) When are the two people at the same height? What is that height?

18. Andrea has three times as many grapes as Hunter. If she gives Hunter six grapes, she will have twice as many as he has now. Write a system of linear equations to model the number of grapes each person has. How many grapes does each person have before the exchange?

Extend

19. Jesse asks his math teacher her age and her husband's age. Jesse's teacher responds that it is not an appropriate question, but that she would use the opportunity to challenge Jesse with a riddle. She says, "One third of my age is ten less than one half of my husband's age. The sum of our ages is 105." How old is Jesse's math teacher?

20. A man swims 200 m against the current of a stream in 3 min. He swims with the same effort downstream for 150 m in 45 s. Create a system of linear equations to determine the man's swimming speed and the speed of the current of the water, in metres per minute.

21. An alloy is a mixture of a metal with a cheaper metal. A jeweller wishes to make pendants using a 94% silver alloy of sterling silver and pure silver. Sterling silver is 92.5% pure silver. The jeweller wishes to make 100 g of the silver alloy. What linear system could be used to determine how many grams of sterling silver and pure silver must be mixed?

22. Eunji wants to spend time at a local amusement park this summer. She is deciding which option will cost less.

Option A: A season's pass costs $22, but she will have to pay $6 for parking each visit.

Option B: A two-visit pass costs $16.50 and includes parking. She can buy as many passes as she wants.

Represent Eunji's options using linear equations. What decision will she need to make before she can choose an option? Why?

Create Connections

23. Gavin leaves Calgary at 1:30 p.m. and drives to Edmonton. He travels at a speed of 110 km/h. At 2:15 p.m., James leaves Calgary in a helicopter travelling along the same route as Gavin. The helicopter travels at 240 km/h. The distance between Calgary and Edmonton is 300 km.

    a) Draw a diagram of their travel or use a table to organize the information.

    b) Describe how to determine whether James will catch up to Gavin.

24. Two bike stores, Bikes-to-Go and Spokz, have different rental options. Spokz charges an hourly rate only. For a 5-h-long rental, both stores charge the same price.

    a) How could you use the graph to determine the hourly rate charged by Spokz?

    b) Write a system of linear equations to model the cost of renting from each store.

    c) How could the graph help you decide which option to choose?