7.4 Parallel and Perpendicular Lines

Controlling the movement of your body is important when you play sports or exercise. The words *parallel* and *perpendicular* describe the position of one thing relative to another. Athletes may need to visualize parallel and perpendicular lines to help them improve their performance. For example,

- A football coach may instruct quarter-backs to position their shoulders perpen-dicular to the target at which they are throwing the ball.
- The gymnast in the photo has trained hard to keep her legs parallel to her arms.

Focus on …

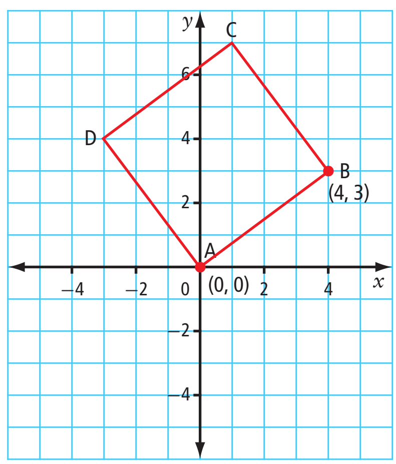
- identifying whether two lines are parallel, perpendicular, or neither
- writing the equation of a line using the coordinates of a point on the line and the equation of a parallel or perpendicular line
- solving problems involving parallel and perpendicular lines

Materials

- two sheets of identical grid paper
- scissors
- ruler

Investigate Slopes of Parallel and Perpendicular Lines

1. On a sheet of grid paper, create a coordinate system by drawing and labelling an x-axis and a y-axis. From another sheet of grid paper, cut out the following three shapes:
   - a square with side length of 5 units
   - a square with side length of 13 units
   - a rectangle with side lengths that are whole units. Draw a diagonal across the rectangle.

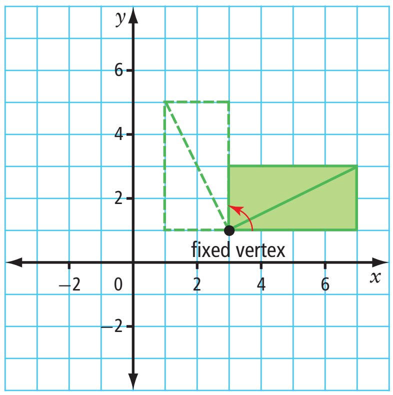


2. For the smaller square, label the vertices A, B, C, and D in a counterclockwise direction. Position the square so that point A is at (0, 0) and point B is at (4, 3). What are the coordinates of vertices C and D? Determine the slope of each side. Compare the slopes of the opposite sides and the adjacent sides.

3. Repeat step 2 using the larger square. Position the square with point A at (0, 0) and B at (12, 5).

4. Position the rectangle on your coordinate system. Determine the slope of each side. Hint: Line up the vertices with the integer coordinates of the grid. Compare the slopes.

5. Compare the slopes of the diagonals formed by rotating the rectangle 90° about one vertex. Discuss your results with a classmate.



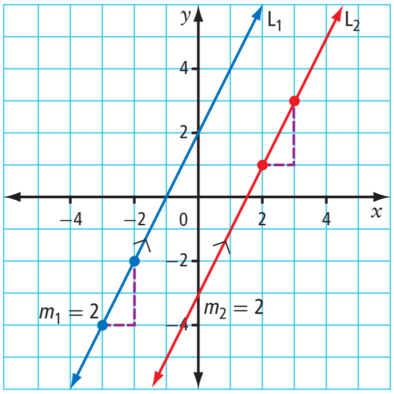
6. **Reflect and Respond** How are the slopes of parallel sides related? Explain using an example.

7. How are the slopes of perpendicular sides related? Explain using an example.

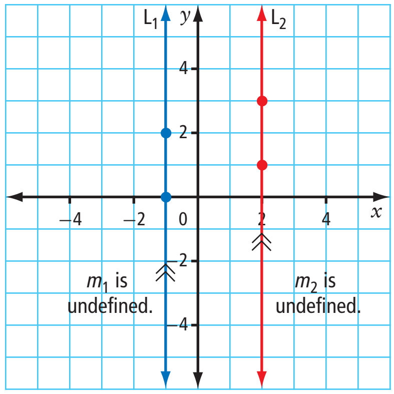
8. a) How are the slopes of vertical sides related?
   b) How are the slopes of a vertical and a horizontal side related?

Link the Ideas

Parallel lines have the same slope but different intercepts. This includes horizontal lines, which have a slope of zero. Vertical lines, which have an undefined slope, are also parallel.



$m_1 = m_2$

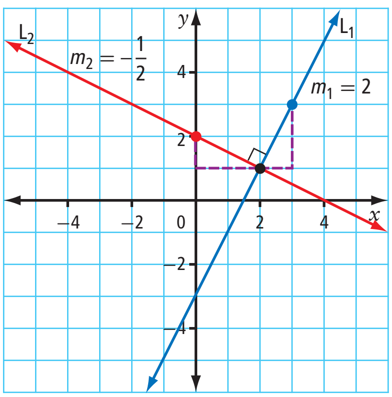


m_1 and m_2 are undefined.

parallel lines

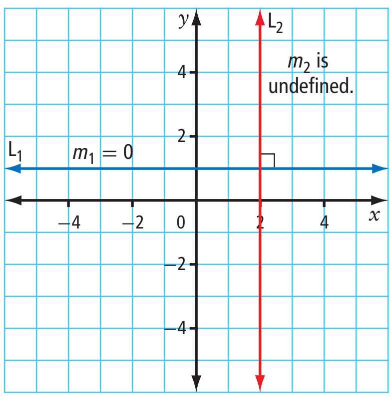
- lines in the same plane that do not intersect
- lines that have the same slope but different intercepts

The slopes of **perpendicular lines** are negative reciprocals of each other. The product of negative reciprocals is −1. A vertical line, which has an undefined slope, and a horizontal line, which has a slope of 0, are perpendicular to each other.



$m_1(m_2) = 2\left(-\frac{1}{2}\right)$

$m_1(m_2) = -1$



$m_1 = 0$ and m_2 is undefined.

perpendicular lines

- two lines that intersect at right angles (90°)
- lines that have slopes that are negative reciprocals of each other.

Example 1 Identify Parallel and Perpendicular Lines

State whether the lines in each pair are parallel, perpendicular, or neither.

a) $y = 3x - 6$
$y = -\frac{1}{3}x + 4$

b) $y = 4x + 3$
$y = 4x - 5$

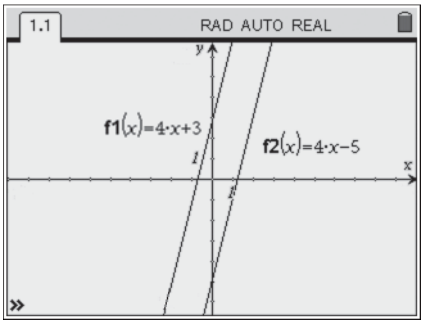
c) $y = 2x + 6$
$6x + 3y + 3 = 0$

Solution

a) The slope of the line $y = 3x - 6$ is 3.
The slope of the line $y = -\frac{1}{3}x + 4$ is $-\frac{1}{3}$.
Since the slopes, 3 and $-\frac{1}{3}$, are negative reciprocals, the lines are perpendicular.

How can you verify that two values are negative reciprocals of each other?

b) The slope of the line $y = 4x + 3$ is 4. The slope of the line $y = 4x - 5$ is also 4. The slopes are equal. So, the lines are parallel.

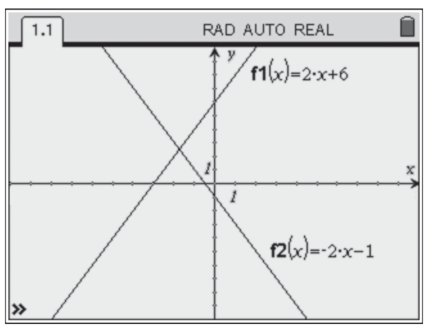


How do you know the lines are not equivalent?

c) The slope of line $y = 2x + 6$ is 2. To determine the slope of the line $6x + 3y + 3 = 0$, rewrite the equation in slope-intercept form, $y = mx + b$.

$$\begin{aligned} 6x + 3y + 3 &= 0 \\ 3y + 3 &= -6x \\ \frac{3y}{3} &= \frac{-6x - 3}{3} \\ y &= -2x - 1 \end{aligned}$$

The slope of the line $y = -2x - 1$ is -2. The slopes 2 and -2 are not equal and they are not negative reciprocals. Therefore, the two lines are neither parallel nor perpendicular.



Your Turn

Determine whether the lines in each pair are parallel, perpendicular, or neither.

a) $y = \frac{1}{2}x - 7$

$y = 2x - 7$

b) $y = 3x - 4$

$y = 3x + \frac{1}{4}$

c) $y = \frac{2}{5}x - 6$

$5x + 2y = 8$

Example 2 Write an Equation Involving a Parallel Line

a) Write the equation of a line that is parallel to $2x - y + 4 = 0$ and through $(1, -6)$. Express the equation in slope-intercept form.

b) Write the equation in general form.

c) Use technology to verify that the lines are parallel.

Solution

a) The slope of the line will be equal to the slope of $2x - y + 4 = 0$. To find the slope, convert $2x - y + 4 = 0$ to slope-intercept form, $y = mx + b$.

$$2x - y + 4 = 0$$
$$2x - y + 4 + y = 0 + y$$
$$2x + 4 = y \text{ or } y = 2x + 4$$

The slope of the line $y = 2x + 4$ is 2.

Method 1: Use Slope-Point Form

Substitute 2 for m and the coordinates of the point $(1, -6)$ for (x_1, y_1).

$$y - y_1 = m(x - x_1)$$
$$y - (-6) = 2(x - 1)$$
$$y + 6 = 2(x - 1)$$

To convert to slope-intercept form, isolate y.

$$y + 6 = 2(x - 1)$$
$$y + 6 = 2x - 2$$
$$y = 2x - 2 - 6$$
$$y = 2x - 8$$

The equation of the line, in slope-intercept form, is $y = 2x - 8$.

Method 2: Use Slope-Intercept Form

The point (1, −6) lies on the line, so the coordinates must satisfy the equation of the line. Substitute 2 for m and the coordinates (1, −6) for (x, y). Then, determine the y-intercept and rewrite the equation.

$$\begin{aligned} y &= mx + b \\ -6 &= 2(1) + b \\ -6 - 2 &= b \\ -8 &= b \end{aligned}$$

Substitute the values for m and b into $y = mx + b$.

$y = 2x + (-8)$

$y = 2x - 8$

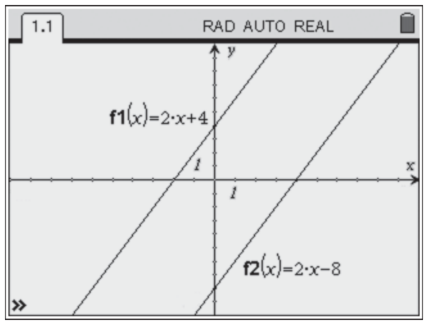
The slope-intercept form of the equation is $y = 2x - 8$.

b) Convert the slope-point equation to general form.

$$\begin{aligned} y + 6 &= 2(x - 1) \\ y + 6 - (y + 6) &= 2(x - 1) - (y + 6) \\ 0 &= 2x - 2 - y - 6 \\ 0 &= 2x - y - 8 \end{aligned}$$

The equation of the line, in general form, is $2x - y - 8 = 0$.

c) The graph to the right shows the original line and the new line, represented by their equations in slope-interecept form.



The equation of original line is $y = 2x + 4$

The equation of original line is $y = 2x - 8$.

The slope-intercept form of the lines and the graph both show that the slopes of the lines are the same, but the y-intercepts are different. Therefore the lines are parallel.

Your Turn

Write the equation of a line that is parallel to $3x + y + 3 = 0$ and passes through (5, −6). Express the equation in slope-intercept form and in general form. Use technology to verify that the lines are parallel.

Example 3 Write an Equation Involving a Perpendicular Line

Write the equation of a line perpendicular to $3x + 2y - 6 = 0$ with an x-intercept of 9. Express the equation in slope-intercept form and in general form.

Solution

To determine the slope of $3x + 2y - 6 = 0$, rewrite the equation in slope-intercept form.

$$\begin{aligned} 3x + 2y - 6 &= 0 \\ 2y &= -3x + 6 \\ y &= -\frac{3}{2}x + 3 \end{aligned}$$

The slope of the line $y = -\frac{3}{2}x + 3$ is $-\frac{3}{2}$.

The negative reciprocal of $-\frac{3}{2}$ is $\frac{2}{3}$.

The reciprocal of $-\frac{3}{2}$ is $-\frac{2}{3}$. So, the negative reciprocal is $-\left(-\frac{2}{3}\right)$ or $\frac{2}{3}$.

Therefore, the slope of a line perpendicular to the given line is $\frac{2}{3}$. Substitute $\frac{2}{3}$ for m and the coordinates of the point (9, 0) for (x_1, y_1) into the slope-point form of an equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= \frac{2}{3}(x - 9) \\ y &= \frac{2}{3}(x - 9) \end{aligned}$$

How else could you find the equation of the line?

For slope-intercept form, $y = mx + b$, expand the slope-point equation.

$$\begin{aligned} y &= \frac{2}{3}(x - 9) \\ y &= \frac{2}{3}x - \overset{3}{\cancel{9}}\left(\frac{2}{\underset{1}{\cancel{3}}}\right) \\ y &= \frac{2}{3}x - 6 \end{aligned}$$

For general form, $Ax + By + C = 0$, rearrange the slope-intercept equation.

$$\begin{aligned} y &= \frac{2}{3}x - 6 \\ 3y &= 2x - 18 \\ 0 &= 2x - 3y - 18 \end{aligned}$$

What other equation could you use to convert to general form? Why might some people choose to use that equation in this solution?

The equation of a line perpendicular to $3x + 2y - 6 = 0$ with an x-intercept of 9 is $y = \frac{2}{3}x - 6$ in slope-intercept form. Written in general form, the equation is $2x - 3y - 18 = 0$.

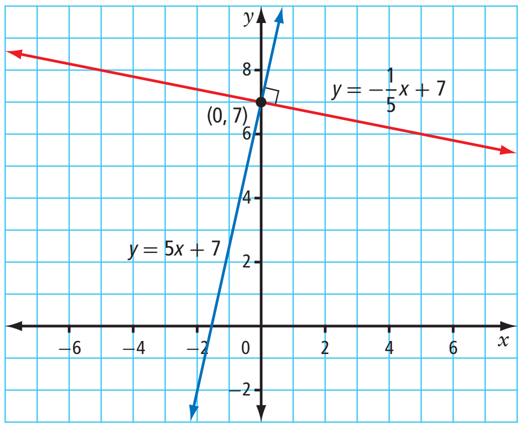
Your Turn

A line is perpendicular to $4x + y - 12 = 0$ and passes through $(8, -6)$. Write the equation of the line in either slope-intercept form or general form.

Key Ideas

- Parallel lines have the same slope and different intercepts. Vertical lines are parallel to each other, as are horizontal lines, if they have different intercepts.
- Perpendicular lines have slopes that are negative reciprocals of each other. A vertical line with an undefined slope and a horizontal line with a slope of zero are also perpendicular.
- The properties of parallel and perpendicular lines can give information about the slopes. Knowing the slopes can help you develop an equation.

  A line perpendicular to $y = 5x + 7$ has the same y-intercept. The line $y = 5x + 7$ has a slope of 5 and a y-intercept of 7. The perpendicular line has a slope of $-\frac{1}{5}$ and a y-intercept of 7. So, the equation of the perpendicular line is $y = -\frac{1}{5}x + 7$.



Check Your Understanding

Practise

1. For a line with each slope, state the slope of a line parallel to it. What is the slope of a line perpendicular to it?

   a) $m = 5$
   b) $m = -7$
   c) $m = -\frac{1}{3}$
   d) $m = \frac{6}{7}$
   e) $m = 0.5$
   f) $m = -0.75$
   g) $m = 0$
   h) m is undefined.

2. State the slopes of lines that are parallel and lines that are perpendicular to each linear equation.

a) $y = \frac{3}{7}x + 4$

b) $y = -x + 9$

c) $3x + y - 5 = 0$

d) $2x + y + 11 = 0$

e) $3x - 2y + 6 = 0$

f) $5x + 4y - 20 = 0$

g) $y = 7$

h) $x + 3 = 0$

3. Consider the line joining points P(−6, 9) and Q(−2, 1).

a) What is the slope of a line parallel to this line?

b) What is the slope of a line perpendicular to this one?

4. For each pair of slopes, what is the value of n if the lines are parallel? What is the value of n if the lines are perpendicular?

a) $\frac{n}{10}, 2$

b) $\frac{24}{n}, -\frac{1}{3}$

c) $\frac{3}{2}, \frac{n}{9}$

d) $\frac{3}{n}, -\frac{7}{2}$

5. Identify whether each pair of lines is parallel, perpendicular, or neither. Explain how you know.

a) $y = -6x$
$y = 6x + 1$

b) $y = \frac{1}{5}x + 3$
$y = -5x - 4$

c) $y = -x + 8$
$x + y = 2$

d) $y = \frac{3}{4}x + 5$
$4x + 3y = 6$

e) $5x + 2y - 10 = 0$
$2x + 5y + 10 = 0$

f) $3x - 4y - 24 = 0$
$4x + 3y - 12 = 0$

6. Write an equation of a line that is parallel to each line and passes through the given point.

a) $y = 2x + 5$, (1, −6)

b) $y = -3x + 7$, (−2, 5)

c) $5x + y - 1 = 0$, (3, −8)

d) $6x - 2y + 10 = 0$, (3, −5)

e) $y = 8$, (3, 4)

f) $x - 5 = 0$, (−1, −8)

7. Write an equation of a line that passes through each point and is perpendicular to each line.

a) $y = 3x + 5$, $(9, 5)$

b) $y = -4x + 7$, $(-12, -7)$

c) $x + 3y + 4 = 0$, $(5, -9)$

d) $4x - 3y - 6 = 0$, $(-2, -1)$

e) $x - 2 = 0$, $(-3, 7)$

f) $y = -5$, $(4, -6)$

Apply

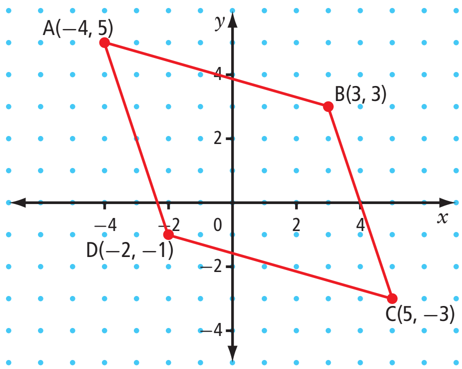
8. Sheldon was asked if line segment AB with A(−9, 2) and B(−3, 4) is parallel to line segment CD with C(−7, −7) and D(1, −3). He sketches a graph of the two line segments and concludes that they appear parallel.

a) Is it correct to assume from a sketch that the two line segments are parallel? Explain.

b) How could you prove that two lines segments are parallel?

c) Is line segment AB parallel to line segment CD? Justify your answer.

9. Is quadrilateral ABCD with vertices A(−4, 5), B(3, 3), C(5, −3), and D(−2, −1) a parallelogram? Justify your answer. Hint: A parallelogram is a quadrilateral with opposite sides parallel.



10. Write the general form equation, $Ax + By + C = 0$, of a line that passes through (7, 5) and is

a) parallel to the x-axis

b) perpendicular to the x-axis

11. Line L_1 passes through points P(n, 4) and Q(1, −2). Line L_2 passes through points R(4, 3) and S(1, 5).

a) What is the value of n if the lines are parallel?

b) If the lines are perpendicular, what is the value of n?

12. Prove that $\triangle ABC$ with vertices A(−3, 5), B(4, 7), and C(−1, −2) is a right triangle.

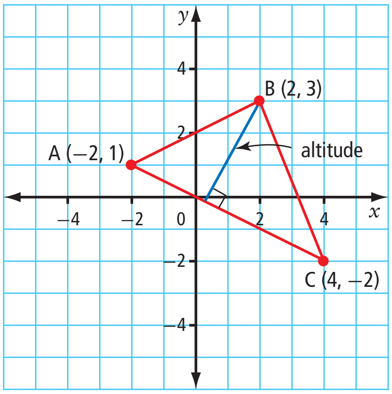
13. Determine an equation representing each line.

a) parallel to $5x + y + 4 = 0$ with a y-intercept of −6

b) perpendicular to $x + 5y - 10 = 0$ with the same y-intercept as $y = 4x - 3$

c) perpendicular to $5x + 4y - 2 = 0$ with the same x-intercept as $3x - 5y = 15$

14. Triangle ABC has vertices A(−2, 1), B(2, 3), and C(4, −2). Write the equation of the line containing the altitude from point B to side AC.

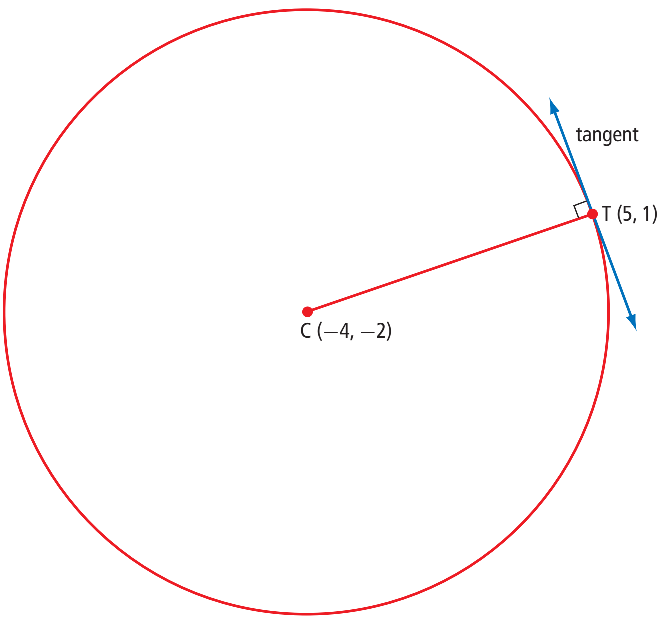


Did You Know?

An altitude is a line segment drawn from a vertex perpendicular to the opposite side.

15. The line through (5, n) and (1, −2) is parallel to the line $3x + 2y - 1 = 0$. What is the value of n?

16. The centre of a circle is located at C(−4, −2) on a coordinate grid. Write the equation of a tangent at point T(5, 1).

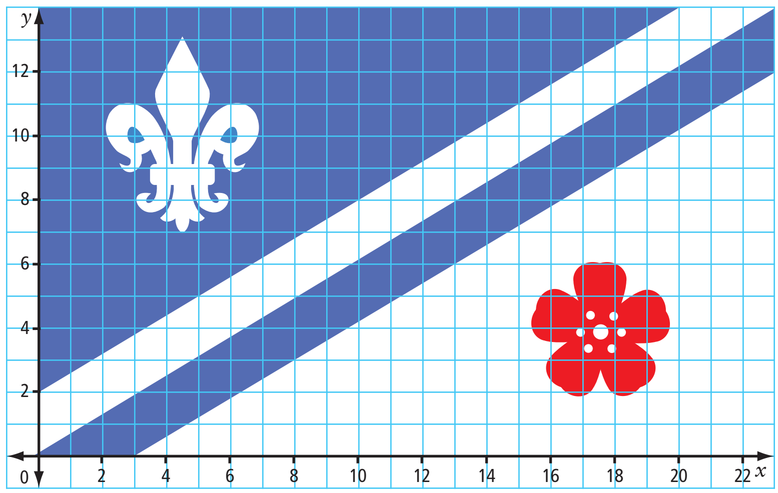


Did You Know?

A tangent is a line that touches a circle at exactly one point. It is perpendicular to the radius at that point.

17. You can monitor your heart rate while you exercise. The Karvonen formula states that the target heart rate during an aerobic workout should be between $H = 0.7(220 - A)$ and $H = 0.8(220 - A)$. In the equations, H represents your target heart rate and A represents your age. Predict whether a graph of these equations would show parallel lines. Justify your answer.

18. In 1982, the French-Canadian Association of Alberta adopted the Franco-Albertan flag. Suppose a coordinate grid is laid over a replica of the flag, with the base along the x-axis and the lower left corner of the flag at the origin. The three parallel lines in the flag pass through the points (20, 14), (23, 14), and (23, 12). Write an equation representing each line.

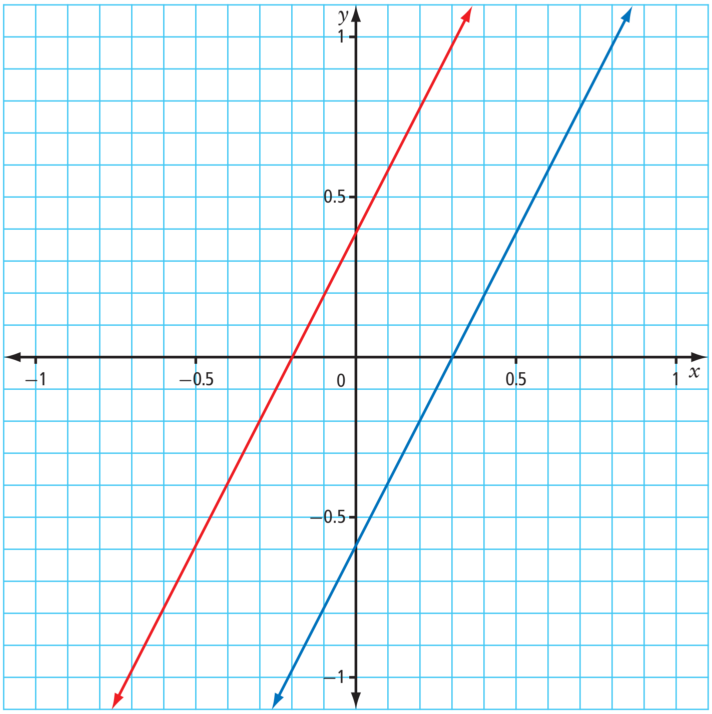


Extend

19. What is the value of n if the graphs of $nx + 4y + 3 = 0$ and $5x - 2y + 6 = 0$ are parallel?

20. The lines $6x - ny + 5 = 0$ and $x + 2y + 4 = 0$ are perpendicular. What is the value of n?

21. What is the shortest distance between the two lines in the graph? Explain your reasoning.



22. Two vertices of right triangle ABC are A(−2, 6) and C(7, 3). If the right angle is at vertex A and vertex B is on the x-axis, identify the coordinates of point B.

23. The lines $nx + 12y - 2 = 0$ and $3x + ny + 6 = 0$ are parallel. What are the possible values of n?

24. Determine the value of n if the lines $nx - 2y + 8 = 0$ and $3x + ny + 6 = 0$ are perpendicular.

Create Connections

25. Is the following statement always true, sometimes true, or never true? "The slopes of perpendicular lines are always negative reciprocals of each other." Explain your reasoning.

26. Suppose you want to determine whether two lines are parallel. Which form of an equation would you prefer to use? Why?