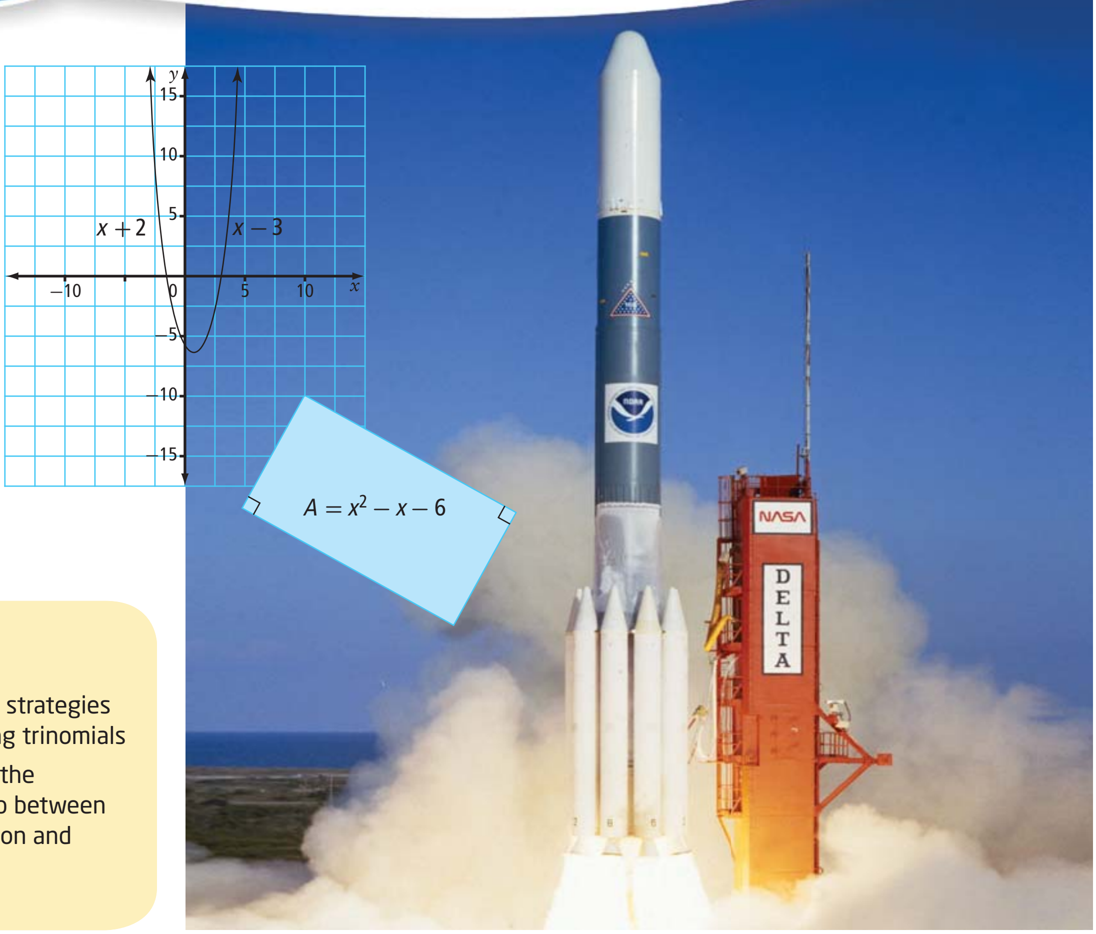
# 5.3 Factoring Trinomials




**Focus on ...**

- developing strategies for factoring trinomials
- explaining the relationship between multiplication and factoring

The trinomial, rectangle, and graph shown all have something in common. They each represent the same relationship. These kinds of relationships allow us to represent physical situations, such as the dimensions of a field or the height of a rocket, with mathematical expressions. Then, the expressions can be used to solve a variety of real-life problems.

**Materials**

- algebra tiles

## Investigate Factoring Trinomials

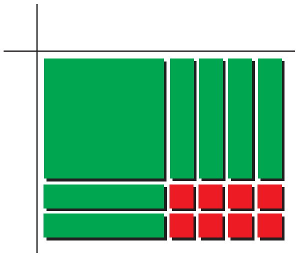
**1. a)** Use algebra tiles to model $(x + 4)(x + 1)$ as the dimensions of a rectangle.

**b)** Complete the rectangle. What is the product of $(x + 4)(x + 1)$?

**c)** How is the product represented by the algebra tiles?

**d)** How are the factors represented by the algebra tiles?

**2. a)** Use algebra tiles to factor the trinomial $x^2 + 6x + 8$. Create a rectangle so that the length and width represent the factors of the trinomial.



**b)** Place tiles along the top and left side of the rectangle to show the length and width of the rectangle. What are the two dimensions?

**c)** Record the dimensions as a product of binomials. What is this product equivalent to?

**d)** Multiply the two binomials. Compare the result to the original trinomial. Are they equivalent?

**3.** Repeat step 2 for each trinomial.

**a)** $x^2 + 5x + 6$

**b)** $x^2 + 8x + 12$

**c)** $x^2 + 3x + 2$

**4.** Each trinomial in step 2 and step 3 is of the form $x^2 + bx + c$. What do you notice about $b$ and $c$ and the binomial factors for each trinomial? Describe the relationship.

**5.** Test your observations from step 4 on each of the following trinomials. Use algebra tiles to check your answer.

**a)** $x^2 + 7x + 6$

**b)** $x^2 + 8x + 15$

**6. Reflect and Respond** Describe a process for finding the factors of a trinomial of the form $x^2 + bx + c$.

## Link the Ideas

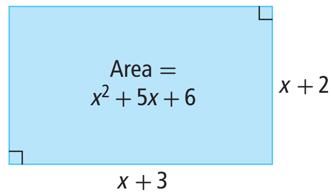
A rectangle can have an area that is a trinomial. By finding the dimensions of the rectangle, you are reversing the process of multiplying two binomials. This process is called *factoring*.

You can factor a trinomial of the form $x^2 + bx + c$ and the form $ax^2 + bx + c$ by studying patterns. Observe patterns that result from multiplying two binomials.

**Factor Trinomials of the Form $ax^2 + bx + c$, $a = 1$**

Multiply $x + 2$ and $x + 3$.

$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 3x + 2x + (2)(3)\\ &= x^2 + 3x + 2x + (2)(3)\\ &= x^2 + (3 + 2)x + (2)(3)\end{aligned}$$



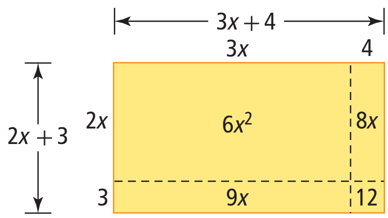

Since $(x + 2)(x + 3) = x^2 + 5x + 6$
and $(x + 2)(x + 3) = x^2 + (3 + 2)x + (2)(3)$,
then $x^2 + 5x + 6 = x^2 + (3 + 2)x + (2)(3)$.

Note that $3 + 2 = 5$ and $(2)(3) = 6$.

To factor trinomials of the form $x^2 + bx + c$, you can use patterns. Replace $bx$ with two terms whose integer coefficients have a sum of $b$ and a product of $c$.

**Factor Trinomials of the Form $ax^2 + bx + c$, $a \neq 1$**

Multiply $2x + 3$ and $3x + 4$.




$$\begin{aligned}(2x + 3)(3x + 4) &= 6x^2 + 8x + 9x + 12\\ &= 6x^2 + 17x + 12\end{aligned}$$

You can combine the two middle terms because they are like terms.

Notice the patterns:
- The sum of 8 and 9 is 17.
- The product of 8 and 9 is the same as (6)(12).

To factor trinomials of the form $ax^2 + bx + c$, you can use patterns. Replace $bx$ with two terms whose integer coefficients have a sum of $b$ and a product of $(a)(c)$.

## Example 1 Factor Trinomials of the Form $ax^2 + bx + c, a = 1$

Factor, if possible.

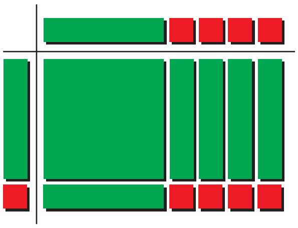
**a)** $x^2 + 5x + 4$
**b)** $x^2 + 4x + 6$
**c)** $x^2 - 29x + 28$
**d)** $x^2 + 3xy - 18y^2$


### Solution

**a)** Factor $x^2 + 5x + 4$.

**Method 1: Use Algebra Tiles**

Arrange one $x^2$-tile, five $x$-tiles, and four 1-tiles into a rectangle. Then, add tiles to show the dimensions.



The dimensions of the rectangle are $x + 4$ and $x + 1$. Therefore, the factors are $x + 4$ and $x + 1$.

How do you know that the dimensions are correct?

**Method 2: Use a Table**

Use a table to find two integers with
- a product of 4
- a sum of 5

In order to have a positive product and a positive sum, what signs do the two integers need to have?

| Factors of 4 | Product | Sum |
|---|---|---|
| 1, 4 | 4 | 5 |
| 2, 2 | 4 | 4 |

Therefore, the factors are $x + 1$ and $x + 4$.

Check:
Multiply.

$$\begin{aligned}(x + 4)(x + 1) &= x(x + 1) + 4(x + 1)\\ &= x^2 + 1x + 4x + 4\\ &= x^2 + 5x + 4\end{aligned}$$

### Did You Know?

When a polynomial cannot be factored such that the factors include only integer coefficients, we say that the polynomial cannot be factored *over the integers*.

**b)** Use a table to find two integers with
- a product of 6
- a sum of 4

In order to have a positive product and a positive sum, what signs do the two integers need to have?

| Factors of 6 | Product | Sum |
|---|---|---|
| 1, 6 | 6 | 7 |
| 2, 3 | 6 | 5 |

No two integers have a product of 6 and sum of 4.
Therefore, you cannot factor $x^2 + 4x + 6$ over the integers.

**c)** Use a table to find two integers with
- a product of 28
- a sum of $-29$

In order to have a positive product and a negative sum, what signs do the two integers need to have?

| Factors of 28 | Product | Sum |
|---|---|---|
| $-1, -28$ | 28 | $-29$ |
| $-2, -14$ | 28 | $-16$ |
| $-4, -7$ | 28 | $-11$ |

Therefore, the factors are $x - 1$ and $x - 28$.

Check:
Multiply.

$$\begin{aligned}(x - 1)(x - 28) &= x(x - 28) - 1(x - 28)\\ &= x^2 - 28x - 1x + 28\\ &= x^2 - 29x + 28\end{aligned}$$

**d)** Use a table to find two integers with
- a product of $-18$
- a sum of 3

In order to have a negative product and a positive sum, what signs do the two integers need to have?

| Factors of $-18$ | Product | Sum |
|---|---|---|
| $1, -18$ | $-18$ | $-17$ |
| $2, -9$ | $-18$ | $-7$ |
| $3, -6$ | $-18$ | $-3$ |
| $6, -3$ | $-18$ | 3 |
| $9, -2$ | $-18$ | 7 |
| $18, -1$ | $-18$ | 17 |

Therefore, the factors are $x + 6y$ and $x - 3y$.
Check:
Multiply.

$$\begin{aligned}(x + 6y)(x - 3y) &= x(x - 3y) + 6y(x - 3y)\\ &= x^2 - 3xy + 6xy - 18y^2\\ &= x^2 + 3xy - 18y^2\end{aligned}$$

## Your Turn

Factor, if possible.

**a)** $x^2 + 7x + 10$

**b)** $r^2 - 10rs + 9s^2$

## Example 2 Factor Trinomials of the Form $ax^2 + bx + c$, $a \neq 1$

Factor, if possible.

**a)** $3x^2 + 8x + 4$

**b)** $6x^2 - 5xy + y^2$

**c)** $3x^2 + 2x + 4$

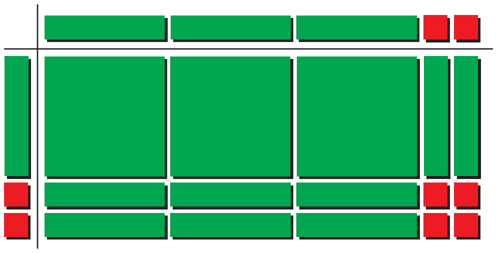
**d)** $24x^2 - 30x - 9$

### Solution

**a)** First, check for a GCF. The GCF of the polynomial $3x^2 + 8x + 4$ is 1.

**Method 1: Use Algebra Tiles**

Arrange three $x^2$-tiles, eight $x$-tiles, and four 1-tiles into a rectangle. Then, add tiles to show the dimensions.



The dimensions of the resulting rectangle are $3x + 2$ and $x + 2$.

How do you know that the dimensions are correct?

Check:

Multiply.

$$\begin{aligned}(3x + 2)(x + 2) &= 3x(x + 2) + 2(x + 2)\\ &= 3x^2 + 6x + 2x + 4\\ &= 3x^2 + 8x + 4\end{aligned}$$

**Method 2: Use a Table**

Use a table to find two integers with

- a product of $(3)(4) = 12$
- a sum of 8

What signs do the two integers need to have?

| Factors of 12 | Product | Sum |
|---|---|---|
| 1, 12 | 12 | 13 |
| 2, 6 | 12 | 8 |
| 3, 4 | 12 | 7 |

Write $8x$ as the sum $2x + 6x$. Then, factor by grouping.

$$\begin{aligned}3x^2 + 8x + 4 &= 3x^2 + (2x + 6x) + 4\\ &= (3x^2 + 2x) + (6x + 4)\\ &= x(3x + 2) + 2(3x + 2)\\ &= (3x + 2)(x + 2)\end{aligned}$$

Therefore, the factors are $3x + 2$ and $x + 2$.

Check:

Multiply.

$$\begin{aligned}(3x + 2)(x + 2) &= 3x(x + 2) + 2(x + 2)\\ &= 3x^2 + 6x + 2x + 4\\ &= 3x^2 + 8x + 4\end{aligned}$$

**b)** First, check for a GCF. The GCF of the polynomial $6x^2 - 5xy + y^2$ is 1. Use a table to find two integers with

- a product of 6
- a sum of $-5$

What signs do the two integers need to have?

| Factors of $6y^2$ | Product | Sum |
|---|---|---|
| $-1, -6$ | 6 | $-7$ |
| $-2, -3$ | 6 | $-5$ |

Write $-5xy$ as $-2xy - 3xy$. Then, factor by grouping.

$$\begin{aligned}6x^2 - 5xy + y^2 &= 6x^2 + (-2xy - 3xy) + y^2\\ &= (6x^2 - 2xy) + (-3xy + y^2)\\ &= 2x(3x - y) - y(3x - y)\\ &= (3x - y)(2x - y)\end{aligned}$$

Therefore, the factors are $3x - y$ and $2x - y$.

Check:

Multiply.

$$\begin{aligned}(3x - y)(2x - y) &= 3x(2x - y) - y(2x - y)\\ &= 6x^2 - 3xy - 2xy + y^2\\ &= 6x^2 - 5xy + y^2\end{aligned}$$

**c)** First, check for a GCF. The GCF of the polynomial $3x^2 + 2x + 4$ is 1. Use a table to find two integers with

- a product of $(3)(4) = 12$
- a sum of 2

What signs do the two integers need to have?

| Factors of 12 | Product | Sum |
|---|---|---|
| 1, 12 | 12 | 13 |
| 2, 6 | 12 | 8 |
| 3, 4 | 12 | 7 |

No two integers have a product of 12 and sum of 2. Therefore, you cannot factor $3x^2 + 2x + 4$ over the integers.

**d)** First, remove the greatest common factor (GCF). The GCF of the polynomial is 3. Therefore, $24x^2 - 30x - 9 = 3(8x^2 - 10x - 3)$. Use a table to find two integers with

- a product of $(8)(-3) = -24$
- a sum of $-10$

What signs do the two integers need to have?

| Factors of $-24$ | Product | Sum |
|---|---|---|
| $-1, 24$ | $-24$ | 23 |
| $-2, 12$ | $-24$ | 10 |
| $-3, 8$ | $-24$ | 5 |
| $-4, 6$ | $-24$ | 2 |
| $-6, 4$ | $-24$ | $-2$ |
| $-8, 3$ | $-24$ | $-5$ |
| $-12, 2$ | $-24$ | $-10$ |
| $-24, 1$ | $-24$ | $-23$ |

Write $-10x$ as $-12x + 2x$. Then, factor by grouping.

$$\begin{aligned} 3(8x^2 - 10x - 3) &= 3(8x^2 - 12x + 2x - 3) \\ &= 3[(8x^2 - 12x) + (2x - 3)] \\ &= 3[4x(2x - 3) + 1(2x - 3)] \\ &= 3(4x + 1)(2x - 3) \end{aligned}$$

Therefore, the factors are 3, $4x + 1$, and $2x - 3$.

Check:
Multiply.

$$\begin{aligned} 3(4x + 1)(2x - 3) &= 3[4x(2x - 3) + 1(2x - 3)] \\ &= 3(8x^2 - 12x + 2x - 3) \\ &= 3(8x^2 - 10x - 3) \\ &= 24x^2 - 30x - 9 \end{aligned}$$

### Your Turn

Factor, if possible.

**a)** $2x^2 + 7x - 4$

**b)** $-3s^2 - 51s - 30$

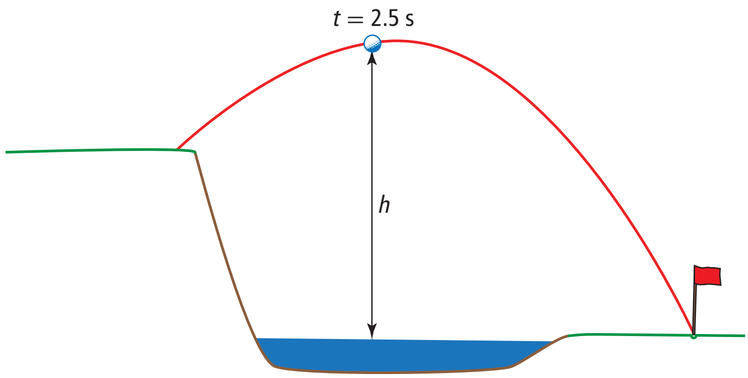
**c)** $2y^2 + 7xy + 3x^2$

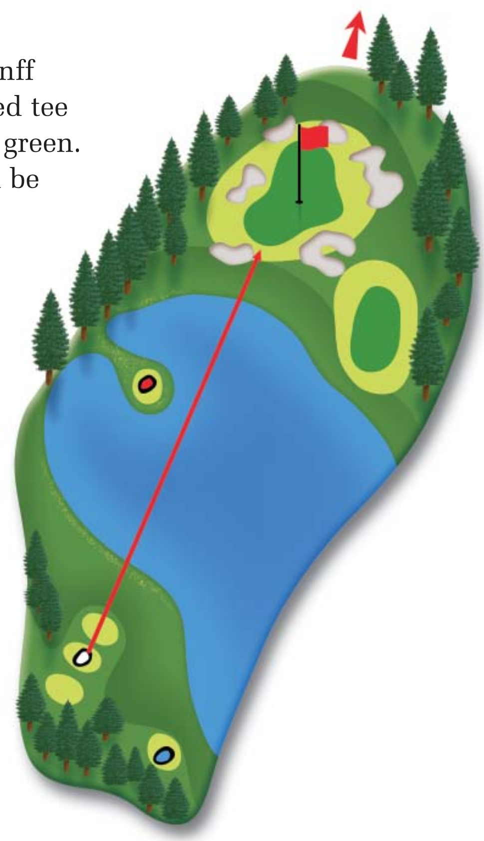
## Example 3 Apply Factoring

The world famous *Devil's Cauldron* is the 4th hole at the Banff Springs Golf Course. This is a tough tee shot from an elevated tee that must carry the ball across a glacial lake to a small bowl green. The approximate height of the ball during a typical shot can be represented by the formula $h = -5t^2 + 25t + 30$, where $t$ is the time, in seconds, and $h$ is the height of the ball relative to the green, in metres.

**a)** Write the formula in factored form.

**b)** What is the height of the golf ball after 2.5 s?



Why is it easier if you remove a GCF of $-5$ instead of $+5$?

**Solution**

**a)** The expression for the height of the golf ball can be factored by first removing the GCF. The GCF of $-5$, 25, and 30 is $-5$.

$-5t^2 + 25t + 30 = -5(t^2 - 5t - 6)$

Use a table to find two integers with

a product of $-6$

a sum of $-5$

| Factors of –6 | Product | Sum |
|---|---|---|
| 1, –6 | –6 | –5 |
| 2, –3 | –6 | –1 |
| 3, –2 | –6 | 1 |
| 6, –1 | –6 | 5 |

Therefore, the factors are $t + 1$ and $t - 6$.

The factored form is $h = -5(t + 1)(t - 6)$.

Check:

Multiply.

$$\begin{aligned} -5(t + 1)(t - 6) &= -5[t(t - 6) + 1(t - 6)] \\ &= -5(t^2 - 6t + t - 6) \\ &= -5(t^2 - 5t - 6) \\ &= -5t^2 + 25t + 30 \end{aligned}$$

**b)** Substitute $t = 2.5$ into $h = -5t^2 + 25t + 30$ or $h = -5(t + 1)(t - 6)$.

$h = -5(2.5)^2 + 25(2.5) + 30$ or $h = -5(2.5 + 1)(2.5 - 6)$

$h = -5(6.25) + 62.5 + 30$ $\quad$ $h = -5(3.5)(-3.5)$

$h = -31.25 + 62.5 + 30$ $\quad$ $h = 61.25$

$h = 61.25$

After 2.5 s, the golf ball is 61.25 m above the green.

## Your Turn

A rescue worker launches a signal flare into the air from the side of a mountain. The height of the flare can be represented by the formula $h = -16t^2 + 144t + 160$. In the formula, $h$ is the height, in feet, above ground, and $t$ is the time, in seconds.

**a)** What is the factored form of the formula?

**b)** What is the height of the flare after 5.6 s?

## Key Ideas

- To factor a trinomial of the form $x^2 + bx + c$, first find two integers with
  - a product of $c$
  - a sum of $b$

  For $x^2 + 12x + 27$, find two integers with
  - a product of 27
  - a sum of 12

  The two integers are 3 and 9.
  Therefore, the factors are $x + 3$ and $x + 9$.

- To factor a trinomial of the form $ax^2 + bx + c$, first factor out the GCF, if possible. Then, find two integers with
  - a product of $(a)(c)$
  - a sum of $b$

  Finally, write the middle term as a sum. Then, factor by grouping.

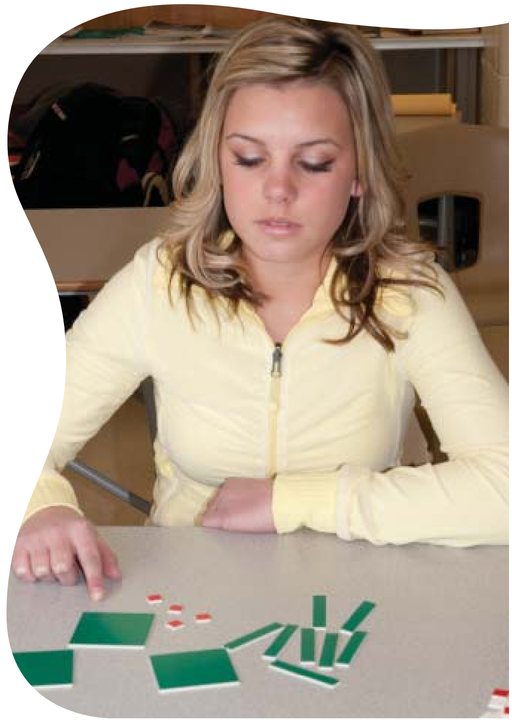
  For $8k^2 - 16k + 6$, the GCF is 2, so
  $8k^2 - 16k + 6 = 2(4k^2 - 8k + 3)$.

  Identify two integers with
  - a product of $(4)(3) = 12$
  - a sum of $-8$

  The two integers are $-2$ and $-6$. Use these two integers to write the middle term as a sum. Then, factor by grouping.

  $2(4k^2 - 2k - 6k + 3) = 2(2k - 3)(2k - 1)$

- You cannot factor some trinomials, such as $x^2 + 3x + 5$ and $3x^2 + 5x + 4$, over the integers.

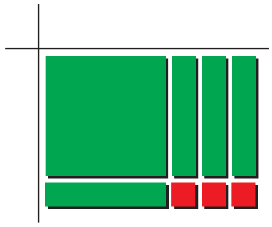


## Check Your Understanding

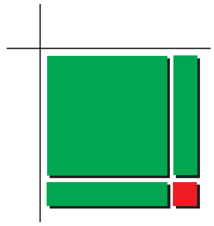
### Practise

**1.** Write the trinomial represented by each rectangle of algebra tiles. Then, determine the dimensions of each rectangle.

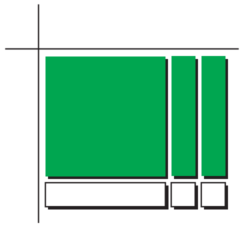
**a)**



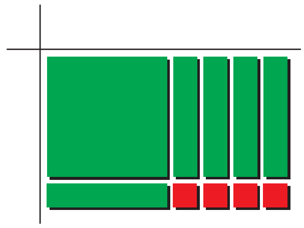
**b)**



**c)**



**d)**



**2.** Use algebra tiles or a diagram to factor each trinomial.

**a)** $2x^2 + 5x + 3$

**b)** $3x^2 + 7x + 4$

**c)** $3x^2 + 7x - 6$

**d)** $6x^2 + 11x + 4$

**3.** Identify two integers with the given product and sum.

**a)** product $= 45$, sum $= 14$

**b)** product $= 6$, sum $= -5$

**c)** product $= -10$, sum $= 3$

**d)** product $= -20$, sum $= -8$

**4.** Factor, if possible.

**a)** $x^2 + 7x + 10$

**b)** $j^2 + 12j + 27$

**c)** $k^2 + 5k + 4$

**d)** $p^2 + 9p + 12$

**e)** $d^2 + 10d + 24$

**f)** $c^2 + 4cd + 21d^2$

**5.** Factor each trinomial.

**a)** $m^2 - 7m + 10$

**b)** $s^2 + 3s - 10$

**c)** $f^2 - 7f + 6$

**d)** $g^2 - 5g - 14$

**e)** $b^2 - 3b - 4$

**f)** $2r^2 - 14rs + 24s^2$

**6.** Factor, if possible.

**a)** $2x^2 + 7x + 5$ **b)** $6y^2 + 19y + 8$

**c)** $3m^2 + 10m + 8$ **d)** $10w^2 + 15w + 3$

**e)** $12q^2 + 17q + 6$ **f)** $3x^2 + 7xy + 2y^2$

**7.** Factor, if possible.

**a)** $4x^2 - 11x + 6$ **b)** $w^2 + 11w + 25$

**c)** $x^2 - 5x + 6$ **d)** $2m^2 + 3m - 9$

**e)** $6x^2 - 3xy - 3y^2$ **f)** $12y^2 + y - 1$

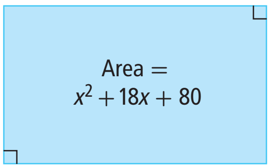
**g)** $6c^2 + 7cd - 10d^2$ **h)** $4k^2 + 15k + 9$

**i)** $a^2 + 11ab + 24b^2$ **j)** $6m^2 + 13mn + 2n^2$

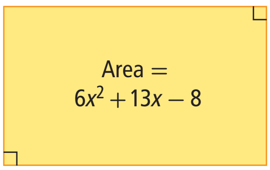
## Apply

**8.** Identify binomials that represent the length and width of each rectangle. Then, calculate the dimensions of the rectangle if $x = 15$ cm.

**a)**




**b)**




**9.** Determine two values of $b$ that allow each expression to be factored.

**a)** $x^2 + bx + 12$ **b)** $y^2 - by + 4$

**c)** $x^2 - bx - 8$ **d)** $p^2 + bp - 10$

**10.** Determine two values of $c$ that allow each expression to be factored.

**a)** $x^2 + 6x + c$ **b)** $a^2 - 8a - c$

**c)** $x^2 - x + c$ **d)** $w^2 + 2w - c$

**11.** Find two values of $n$ that allow each trinomial to be factored over the integers.

**a)** $x^2 + nx + 16$

**b)** $3y^2 + ny + 25$

**c)** $6a^2 + nab + 7b^2$

12. Determine one value of $k$ that allows each trinomial to be factored over the integers.

a) $36m^2 + 18m + k$

b) $18x^2 - 15x + k$

c) $kp^2 - 18pq + 16q^2$

13. a) Make up an example of a trinomial expression that cannot be factored.

b) Explain why it cannot be factored.

14. **Unit Project** Use algebra tiles or area models to show the following relationships. Create a poster displaying your models.

a) the relationship between a monomial multiplied by a binomial and common factoring

b) the relationship between a binomial multiplied by a binomial and factoring a trinomial of the form $ax^2 + bx + c$, where $a$, $b$, and $c$ are integers

15. You can estimate the height, $h$, in metres, of a toy rocket at any time, $t$, in seconds, during its flight. Use the formula $h = -5t^2 + 23t + 10$. Write the formula in factored form. Then, calculate the height of the rocket 3 s after it is launched.

16. The total revenue from sales of ski jackets can be modelled by the expression $720 + 4x - 2x^2$, where $x$ represents the number of jackets sold above the minimum needed to break even. Revenue is also calculated as the product of the number of jackets sold and the price per jacket. Factor the given expression to determine the number sold and the price per jacket. The minimum price of a jacket is \$18.
Hint: As the price increases, the number sold decreases.

## Extend

**17.** Find three values of $k$ such that the trinomial $3x^2 + kx + 5$ can be factored over the integers.

**18.** A square has an area of $9x^2 + 30xy + 25y^2$ square centimetres. What is the perimeter of the square? Explain how you determined your answer.

**19.** You have been asked to factor the expression $30x^2 - 39xy - 9y^2$ Explain how you would factor this expression. What are the factors?

**20.** The area of a certain shape can be represented by the expression $8x^2 + 10x - 7$.

**a)** Identify a possible shape.

**b)** Write expressions for the possible dimensions of the shape you identified in part a).

## Create Connections

**21.** Describe, using examples, how multiplying binomials and factoring a trinomial are related.

**22.** **Unit Project**

**a)** Use algebra tiles to create a model of a polynomial of your choice.

**b)** Create a piece of art that includes your polynomial in some way. Your artwork may be a drawing, painting, sculpture, or other form of your choice.