4.4 Irrational Numbers

Focus on ...

- representing, identifying, and simplifying irrational numbers
- converting between powers with rational exponents and radicals
- converting between mixed radicals and entire radicals
- solving problems involving radicals

A golden rectangle has sides that are in a ratio that is pleasing to the eye. The ratio of the length to the width in a golden rectangle is called the golden ratio. Many artists use the golden ratio when composing their paintings. This painting titled *Coming Rain* by Ayla Bouvette appears to have several golden rectangles.

The golden ratio is an **irrational number**. Irrational numbers are often called artistic numbers, since they appear in art, architecture, nature, and geometry.

irrational number

- a number that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers, and $b \neq 0$
- cannot be expressed as a terminating or repeating decimal
- $\pi = 3.1415...$ $\sqrt{5} = 2.236...$

Did You Know?

Ayla Bouvette is an artist with Okanagan, Anishnaabe, and Red River Métis heritage. Her work uses the themes and designs of her heritage. She sells prints of her works to support local powwow fundraising.

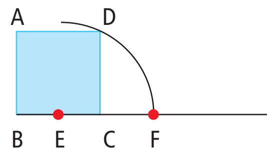
Investigate the Golden Rectangle

Unit Project

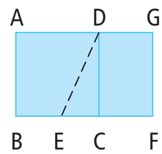
Materials
- ruler
- compasses

1. Draw a square on a blank sheet of paper. Label the vertices of the square ABCD. Measure and record the side length of the square.

2. Complete the following steps.
- Mark the midpoint of side BC as E.
- Extend line BC so that it is about double in length.
- Use compasses to draw an arc with radius DE so that it intersects line BC at point F.



3. Complete the golden rectangle by drawing DG and FG.



4. a) Calculate the length of DE to four decimal places.

b) Measure the length of DE. How does the actual measurement compare to your calculated value?

c) Calculate the length of BF to four decimal places. Hint: DE is the same length as EF.

d) Measure the length of BF. How does the actual measurement compare to your calculated value?

5. Reflect and Respond

a) The ratio of the length to the width in a golden rectangle is called the golden ratio. Write an exact expression for the golden ratio.

b) What is the approximate value of the golden ratio, to two decimal places?

c) In the painting on page 184 describe the golden rectangles you see. Discuss your ideas with a classmate.

6. a) Look for three rectangular shapes in the classroom that you think may be in the golden ratio. Use a table to organize your findings.
- Measure the length and width of each shape.
- Calculate the ratios of the sides as you did for the rectangle you drew.

b) How do the ratios compare? How close were the rectangles you chose to golden rectangles?

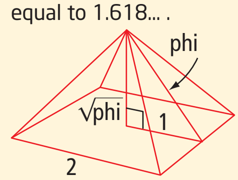
c) Compare your results with those of a classmate.

WWW Web Link

For more information about the golden ratio and the golden rectangle, go to www.mhrmath10.ca and follow the links.

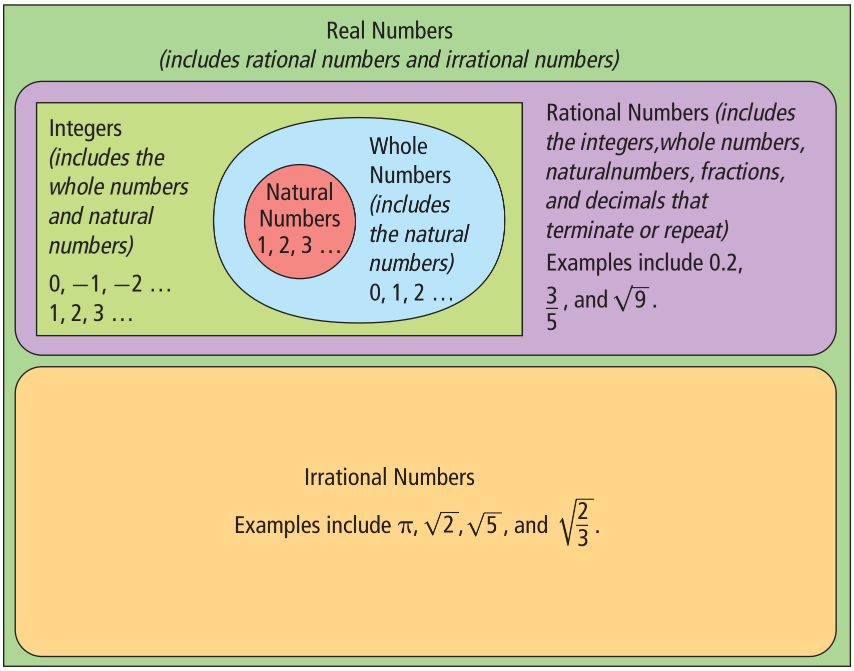
Did You Know?

The Rhind papyrus of Egypt records the building of the Great Pyramid of Giza in 4700 B.C.E. The pyramid has proportions according to the golden ratio. The golden ratio is often represented as *phi,* or ϕ. This is approximately equal to 1.618... .



Link the Ideas

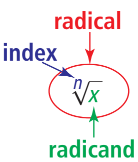
The rational numbers include the natural numbers, whole numbers, and integers. These numbers and the irrational numbers form a set called the real numbers.



What subsets do integers belong to? whole numbers? natural numbers?

radical

- consists of a root symbol, an index, and a radicand



- can be rational, $\sqrt{4}$, or irrational, $\sqrt{2}$

radicand

- the quantity under the radical sign

index

- indicates what root to take

Powers with fractional exponents can be written as **radicals** in the form $x^{\frac{1}{n}} = \sqrt[n]{x}$, where $n \neq 0$. When n is even, x cannot be negative, since the product of an even number of equal factors is always positive.

For example, $\left(3^{\frac{1}{2}}\right)\left(3^{\frac{1}{2}}\right) = 3^{\left(\frac{1}{2} + \frac{1}{2}\right)}$
$= 3$

You know that $(\sqrt{3})(\sqrt{3}) = 3$.

Therefore, $3^{\frac{1}{2}} = \sqrt{3}$.

A power can be expressed as a radical in the form

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = (\sqrt[n]{x})^m$$

or

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

where m and n are integers.

For example, $2^{\frac{3}{4}}$ can be written as $\sqrt[4]{2^3}$ or $\left(\sqrt[4]{2}\right)^3$.

A fractional exponent can be written in decimal form.

For example, $\sqrt[5]{6^3} = 6^{\frac{3}{5}}$
$= 6^{0.6}$

If the radicand is a number, you can evaluate a power with a fractional or decimal exponent.

For example, $\sqrt{5^3} = 5^{\frac{3}{2}}$
$= 11.1803\ldots$

When the index is 2, it is commonly not written.

Did You Know?

The most famous irrational number is likely pi. The Babylonians (about 2000 to 1600 B.C.E.) were the first to approximate pi to a value of 3. Since that time, pi has been calculated to 1 241 100 000 000 decimal places.

The value of pi is 3.14159265368979323846264338327 95...

Example 1 Convert From a Power to a Radical

Express each power as an equivalent radical.

a) $64^{\frac{1}{2}}$

b) $16^{\frac{3}{4}}$

c) $(8x^2)^{\frac{1}{3}}$

Solution

Write each power as a radical.

Use the denominator of the exponent as the index of the radical.

a) $64^{\frac{1}{2}} = \sqrt{64}$

b) $16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3$

c) $(8x^2)^{\frac{1}{3}} = \sqrt[3]{8x^2}$

Your Turn

Express each power as a radical.

a) $10^{\frac{1}{4}}$

b) $1024^{\frac{1}{3}}$

c) $(x^4)^{\frac{3}{8}}$

Example 2 Convert From a Radical to a Power

Express each radical as a power with a rational exponent.

a) $\sqrt[4]{4^3}$ **b)** $\sqrt[5]{3^4}$ **c)** $\sqrt{s^3}$

Solution

Write each radical as a power.
Use the index as the denominator of the exponent.

a) $\sqrt[4]{4^3} = (4^3)^{\frac{1}{4}}$
$= 4^{\frac{3}{4}}$
$= 4^{0.75}$

b) $\sqrt[5]{3^4} = 3^{\frac{4}{5}}$

c) $\sqrt{s^3} = (s^3)^{\frac{1}{2}}$
$= s^{\frac{3}{2}}$

Your Turn

Express each radical as a power.

a) $\sqrt{125}$ **b)** $\sqrt[3]{y^5}$ **c)** $\sqrt[n]{27^2}$

mixed radical

- the product of a rational number and a radical
- for example, $3\sqrt{2}$, and $\frac{1}{2}\sqrt[3]{6}$

entire radical

- the product of 1 and a radical
- for example, $\sqrt{32}$, and $\sqrt[3]{2^5}$

Example 3 Convert Mixed Radicals to Entire Radicals

Express each **mixed radical** as an equivalent **entire radical**.

a) $5\sqrt{11}$

b) $2\sqrt[3]{5}$

c) $1.5\sqrt[3]{6}$

Solution

a) $5\sqrt{11} = \sqrt{(5^2)}\sqrt{(11)}$
$= \sqrt{(5^2)(11)}$
$= \sqrt{(25)(11)}$
$= \sqrt{275}$

What is the index?
How does it help you convert to an entire radical?

b) $2\sqrt[3]{5} = \sqrt[3]{(2^3)}\sqrt[3]{(5)}$
$= \sqrt[3]{(2^3)(5)}$
$= \sqrt[3]{(8)(5)}$
$= \sqrt[3]{40}$

c) $1.5\sqrt[3]{6} = \sqrt[3]{(1.5^3)}\,\sqrt[3]{(6)}$ or $\frac{3}{2}\sqrt[3]{6} = \sqrt[3]{\left(\frac{3}{2}\right)^3}\,\sqrt[3]{(6)}$

$$= \sqrt[3]{(1.5^3)(6)}$$
$$= \sqrt[3]{(3.375)(6)}$$
$$= \sqrt[3]{20.25}$$

$$= \sqrt[3]{\left(\frac{3}{2}\right)^3(6)}$$
$$= \sqrt[3]{\left(\frac{27}{8}\right)(6)}$$
$$= \sqrt[3]{\frac{81}{4}}$$

What could you do to express the radical in lowest terms?

Your Turn

Convert each mixed radical to an equivalent entire radical.

a) $9\sqrt[3]{4}$ **b)** $4.2\sqrt{18}$ **c)** $\frac{1}{2}\sqrt{10}$

Example 4 Convert Entire Radicals to Mixed Radicals

Express each entire radical as an equivalent mixed radical.

a) $\sqrt{27}$ **b)** $\sqrt{50}$ **c)** $\sqrt{48}$ **d)** $\sqrt[4]{80}$

Solution

a) $\sqrt{27} = \sqrt{(9)(3)}$
$$= \sqrt{9}\,\sqrt{3}$$
$$= 3\sqrt{3}$$

What value is a perfect square? How does this help you?

b) $\sqrt{50} = \sqrt{(25)(2)}$
$$= \sqrt{25}\,\sqrt{2}$$
$$= 5\sqrt{2}$$

c) $\sqrt{48} = \sqrt{(16)(3)}$
$$= \sqrt{16}\,\sqrt{3}$$
$$= 4\sqrt{3}$$

or $\sqrt{48} = \sqrt{(4)(12)}$
$$= \sqrt{4}\,\sqrt{12}$$
$$= 2\sqrt{(4)(3)}$$
$$= 2\sqrt{4}\,\sqrt{3}$$
$$= 2(2)\sqrt{3}$$
$$= 4\sqrt{3}$$

d) $\sqrt[4]{80} = \sqrt[4]{(2)(2)(2)(2)(5)}$
$$= \sqrt[4]{(2^4)(5)}$$
$$= 2\sqrt[4]{5}$$

How does prime factorization help you?

Your Turn

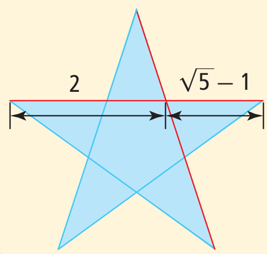
Convert each entire radical to an equivalent mixed radical.

a) $\sqrt{40}$ **b)** $\sqrt{108}$ **c)** $\sqrt[3]{32}$

Did You Know?

The pentagram is also called the star polygon. Inside a pentagram is a pentagon.

Each length of a pentagram intersects two other lengths of the pentagram. The intersection points divide each length according to the golden ratio.



Web Link

For more information about pentagrams and how to draw one, go to www.mhrmath10.ca and follow the links.

Example 5 Order Irrational Numbers

Order these irrational numbers from least to greatest.

$2\sqrt{18}$ $\sqrt{8}$ $3\sqrt{2}$ $\sqrt{32}$

Solution

Method 1: Express Each Irrational Number as an Entire Radical

$$\begin{aligned} 2\sqrt{18} &= \sqrt{(2^2)(18)} \\ &= \sqrt{72} \end{aligned}$$

$$\sqrt{8} = \sqrt{8}$$

$$\begin{aligned} 3\sqrt{2} &= \sqrt{(3^2)(2)} \\ &= \sqrt{18} \end{aligned}$$

$$\sqrt{32} = \sqrt{32}$$

The radicals in order from least to greatest are
$\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$, and $\sqrt{72}$
or
$\sqrt{8}$, $3\sqrt{2}$, $\sqrt{32}$, and $2\sqrt{18}$.

In this case, how could you have used mixed radicals to order the irrational numbers?

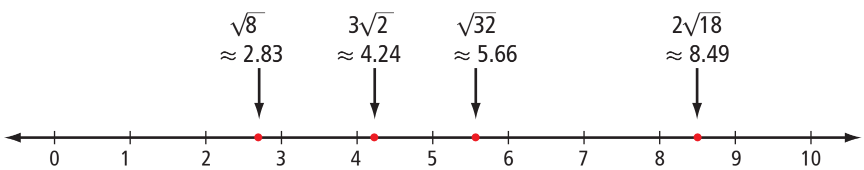
Method 2: Estimate the Approximate Values and Plot on a Number Line

Estimate the value of each radical.

$$\begin{aligned} 2\sqrt{18} &= (2)(4.243\ldots) \\ &= 8.485\ldots \\ &\approx 8.49 \end{aligned} \qquad \begin{aligned} \sqrt{8} &= 2.828\ldots \\ &\approx 2.83 \end{aligned}$$

$$\begin{aligned} 3\sqrt{2} &= (3)(1.414\ldots) \\ &= 4.243\ldots \\ &\approx 4.24 \end{aligned} \qquad \begin{aligned} \sqrt{32} &= 5.657\ldots \\ &\approx 5.66 \end{aligned}$$

Plot the approximations on a number line.



Using the approximations on the number line, the radicals in order from least to greatest are $\sqrt{8}$, $3\sqrt{2}$, $\sqrt{32}$, and $2\sqrt{18}$.

Your Turn

Use two different methods to order the following irrational numbers from greatest to least: $2\sqrt{54}$, $\sqrt{192}$, $5\sqrt{10}$.

Example 6 Solve Problems Involving Irrational Numbers

The Seabee Mine is located at Laonil Lake, SK. In 2007, the mine produced a daily average of gold great enough to fill a cube with a volume of 180 cm^3. If five days of gold production is cast into a cube, what is its edge length?

Solution

The volume of gold produced in five days is (5)(180) = 900 cm^3. The formula for the volume, V, of a cube is $V = s^3$, where s is the length of one side. Substitute 900 for the volume. Solve for s by taking the cube root of both sides of the equation.

Alternatively, you can raise both sides to the exponent $\frac{1}{3}$.

$$\begin{aligned} 900 &= s^3 \\ \sqrt[3]{900} &= \sqrt[3]{s^3} \\ 9.654\ 89\ldots &= s \end{aligned} \qquad \text{or} \qquad \begin{aligned} 900 &= s^3 \\ 900^{\frac{1}{3}} &= (s^3)^{\frac{1}{3}} \\ 9.654\ 89\ldots &= s \end{aligned}$$

The edge of the cube would be approximately 9.7 cm long.

Your Turn

Assume the Seabee Mine doubles its daily gold production to 360 cm^3. What is the edge length of a cube of gold produced in a five-day period?

Did You Know?

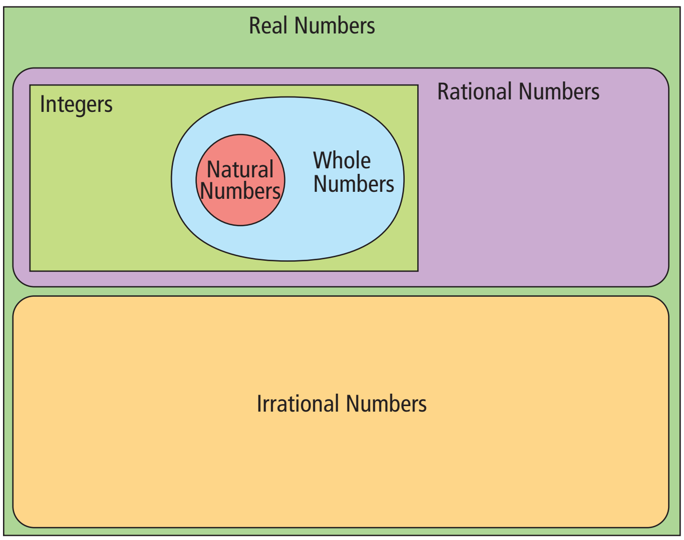
To date, the total amount of gold that has been extracted from Earth would fill only two Olympic sized swimming pools.

Did You Know?

It can be profitable to extract gold from ore grades as low as 0.5 g per 1000 kg of ore. This grade of ore is so low in gold that the gold is not visible. Ore grades of about 30 g per 1000 kg are needed before you can see gold.

Key Ideas

- Rational numbers and irrational numbers form the set of real numbers.



- Radicals can be expressed as powers with fractional exponents.

  $$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

  The index of the radical has the same value as the denominator of the fractional exponent.

  $\sqrt[3]{10} = 10^{\frac{1}{3}}$ $\qquad$ $\sqrt[5]{7^3} = 7^{\frac{3}{5}}$

- Radicals can be entire radicals such as $\sqrt{72}$, $\sqrt[5]{96}$, and $\sqrt[3]{\frac{54}{8}}$. They can also be mixed radicals such as $6\sqrt{2}$, $2\sqrt[5]{3}$, and $\frac{3\sqrt{2}}{2}$. You can convert between entire radicals and mixed radicals.

- You can order radicals that are irrational numbers using different methods:
  - Use a calculator to produce approximate values.
  - Express each irrational number as an entire radical.

Check Your Understanding

Practise

1. Express each power as an equivalent radical.

a) $4^{\frac{3}{2}}$ **b)** $32^{\frac{1}{5}}$ **c)** $64^{0.5}$

d) $\left(\frac{1}{100}\right)^{\frac{1}{4}}$ **e)** $\left(\frac{y^4}{x^3}\right)^{\frac{1}{3}}$ **f)** $(m^n)^{\frac{3}{2}}$

2. Express each radical as a power.

a) $\sqrt{(12p)^3}$ **b)** $\sqrt[5]{5^3}$ **c)** $\sqrt[4]{x^3}$

d) $\sqrt[3]{\frac{s^3}{t^5}}$ **e)** $\sqrt{y^{\frac{5}{3}}}$ **f)** $\sqrt[n]{8}$

3. Evaluate each expression. State the result to four decimal places, if necessary.

a) $\sqrt{0.36}$ **b)** $(27)^{\frac{1}{3}}$ **c)** $4\sqrt{17}$

d) $(65)^{\frac{2}{3}}$ **e)** $0.3(22)^{\frac{1}{2}}$ **f)** $\frac{\sqrt{36}}{\sqrt{7}}$

4. Express each mixed radical as an equivalent entire radical.

a) $3\sqrt{11}$ **b)** $7\sqrt{2}$ **c)** $3\sqrt{5}$

d) $2\sqrt{7}$ **e)** $3\sqrt{3}$ **f)** $10\sqrt{6}$

5. Express each mixed radical as an equivalent entire radical.

a) $2\sqrt[3]{7}$ **b)** $3\sqrt[3]{3}$ **c)** $10\sqrt[3]{5}$

d) $4\sqrt[3]{2}$ **e)** $3\sqrt[4]{2}$ **f)** $2\sqrt[4]{5}$

6. Express each entire radical as an equivalent mixed radical.

a) $\sqrt{12}$ **b)** $\sqrt{50}$ **c)** $\sqrt{48}$

d) $\sqrt{72}$ **e)** $\sqrt{45}$ **f)** $\sqrt{500}$

7. Express each entire radical as an equivalent mixed radical.

a) $\sqrt[3]{24}$ **b)** $\sqrt[3]{54}$ **c)** $\sqrt[3]{243}$

d) $\sqrt[3]{40}$ **e)** $\sqrt[4]{32}$ **f)** $\sqrt[4]{243}$

8. Order each set of numbers from least to greatest. Then, identify the irrational numbers.

a) $\frac{5}{8}$ $\quad 0.\overline{6}$ $\quad \sqrt{0.25}$ $\quad \sqrt[3]{0.84}$

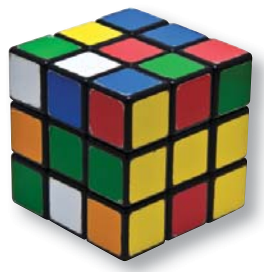
b) $3\sqrt{28}$ $\quad \sqrt{225}$ $\quad 15\frac{4}{5}$ $\quad \sqrt[4]{625}$

9. Plot each set of numbers on a number line. Which of the numbers in each set is irrational?

a) $3\sqrt{4}$ $\quad 6.\overline{6}$ $\quad \sqrt{39}$ $\quad \sqrt[3]{515}$

b) $4\frac{3}{11}$ $\quad \sqrt[3]{125}$ $\quad \frac{4\sqrt{125}}{5}$ $\quad 3\sqrt{8}$

10. The Rubik's Cube is a mechanical puzzle. Calculate the edge length of a Rubik's Cube with a volume of 38.44 cm^3, to three decimal places.



11. Pacific halibut are the largest of all flatfish. The relationship between the length and mass of Pacific halibut can be approximated using the equation $l = 0.46\sqrt[3]{m}$. In this equation, l is the length, in metres, and m is the mass, in kilograms. Use the equation to predict the length of a 25-kg Pacific halibut.

Did You Know?

The range of the Pacific halibut extends along the Pacific coast to the Bering Sea. These fish are important to coastal First Nations who harvest them for food and ceremonial purposes.

Apply

12. Police can estimate the speed of a car by the length of the skid marks made when the driver braked. The formula is $v = \sqrt{30df}$. In this formula, v is the speed, in miles per hour, d is the length of the skid marks, in feet, and f is the coefficient of friction. What was the speed of a vehicle if the skid marks were 75 ft long and the coefficient of friction was 0.7?

13. **Unit Project** Christina is a weaver in Pangnirtung, NU. The dimensions of the tapestry that she is working on represent the golden ratio.

a) If the longer dimension of the tapestry is 60 cm, what is the shorter dimension? Express the answer to the nearest hundredth of a centimetre.

b) What is the total area of the tapestry?

Did You Know?

The velocity of a satellite in *geosynchronous orbit* around Earth matches the rotation of Earth. Since the orbital velocity matches Earth's rotation, the satellite appears to stay in one spot over Earth. However, the satellite is actually travelling at more than 11 000 km/h.

14. When a satellite is h kilometres above Earth, the time, t, in minutes, to complete one orbit is given by the formula

$$t = \frac{\sqrt{(6370 + h)^3}}{6024}.$$

a) A telecommunications satellite is placed 30 km above Earth. How long does it take the satellite to make one orbit?

b) A satellite is placed in geosynchronous orbit about Earth. What must its altitude be?

WWW Web Link

For information about Canada's satellites, go to www.mhrmath10.ca and follow the links.

15. The formula $l = \frac{8t^2}{\pi^2}$ represents the swing of a pendulum. In this formula, l is the length of the pendulum, in feet, and t is the time, in seconds, it takes to swing back and forth once. What is the length of a pendulum that makes one swing in 2 s?

16. An electronics store owner researched the number of customers who would attend a limited-time sale. She modelled the relationship between the sales discount and the length of the sale using the formula $N = 580\sqrt[3]{Pt}$. In this formula, N is the number of customers expected, P is the percent of the sales discount, and t is the number of hours of the sale. What sales discount should the store offer in order to attract 500 shoppers in 8 h?

Kira made a start to the solution. Complete her work.

$$N = 580\sqrt[3]{Pt}$$

$$500 = 580\sqrt[3]{P(8)}$$

$$\frac{500}{580} = \frac{580}{580}\sqrt[3]{8P}$$

17. The amount of current, I, in amperes, that an appliance uses can be calculated by the formula $I = \left(\frac{P}{R}\right)^{\frac{1}{2}}$, where P is the power, in watts, and R is the resistance, in ohms. How much current does an appliance use if $P = 120$ W and $R = 3\ \Omega$? Express your answer to one decimal place.

18. **Unit Project** Many aspects of nature, such as the spiral patterns of leaves and seeds, can be described using the Fibonacci sequence. The sequence is 1, 1, 2, 3, 5, 8, 13, … . The expression for the nth term of the Fibonacci sequence is called Binet's formula. The formula is $F_n = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n$.
Use Binet's formula to find F_3.

WWW Web Link

Pine cones, which grow in spirals, show the Fibonacci spirals. Try to locate spiral patterns in the pine cone shown. To learn more about Fibonnaci spirals, go to www.mhrmath10.ca and follow the links.

Extend

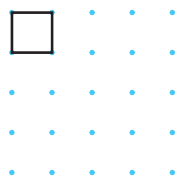
19. Express as a power with a rational exponent.

a) $\sqrt{\sqrt{2^{\frac{4}{5}}}}$ **b)** $\sqrt[4]{\sqrt{256}}$

20. Is the statement $\sqrt[4]{(-x)^4} = x$ sometimes, always, or never true? Explain your reasoning.

Create Connections

21. A 1-by-1 square can be drawn on 5-by-5 dot paper as shown.



a) Draw as many different sized squares as possible on a piece of 5-by-5 dot paper. How many did you find?

b) What is the side length of each square? Which side lengths are rational and which are irrational?

c) What is the area of each square?

22. Describe the relationship between a radical and its equivalent power with a rational exponent.

23. Research the history of up to three algebraic or mathematical symbols of your choice. You might consider the radical sign, pi, phi, or zero. For each symbol, explain who developed it and why they created it.

24. **Unit Project** Use what you have learned about radicals to analyse the golden ratio. Use the following methods as a guide.
- Make a timeline about the history of the golden ratio.
- Explain the exact relationship between the dimensions of the golden rectangle and the golden ratio.
- Use a visual to help describe one other example of the golden ratio.