4.3 Rational Exponents

Focus on ...

- applying the exponent laws to expressions using rational numbers or variables as bases and rational exponents
- solving problems that involve powers with rational exponents

On a piano keyboard, the pitches of any two adjacent keys are related by a ratio equal to $\sqrt[12]{2}$. This is defined as the number that, when multiplied by itself 12 times, results in 2. You may have noticed that there is no $\sqrt[12]{}$ key on your calculator. How can a piano technician evaluate this number?

Roots other than the square root often occur in science, technology, music, art, and other disciplines. How can you represent such roots in a way that makes them easy to work with?

Investigate Rational Exponents

1. According to the product rule for powers

$$\begin{aligned}\left(9^{\frac{1}{2}}\right)\left(9^{\frac{1}{2}}\right) &= 9^{\frac{1}{2}+\frac{1}{2}} \\ &= 9^1 \\ &= 9\end{aligned}$$

You can reverse these statements to get

$$\begin{aligned}9^1 &= 9^{\frac{1}{2}+\frac{1}{2}} \\ &= \left(9^{\frac{1}{2}}\right)\left(9^{\frac{1}{2}}\right)\end{aligned}$$

What is the value of $9^{\frac{1}{2}}$? Check your answer with a calculator.

2. Predict values for $4^{\frac{1}{2}}$, $16^{\frac{1}{2}}$, $36^{\frac{1}{2}}$, and $49^{\frac{1}{2}}$. Use a calculator to check your predictions. Were you correct?

3. Predict the value of $8^{\frac{1}{3}}$. Explain your thinking. Check your prediction.

4. Reflect and Respond

a) Explain how determining $49^{\frac{1}{2}}$ and your definition for square root are related.

b) Express the 12th root of 2 as a power. Evaluate using your calculator. Express the answer to six decimal places.

c) Use your calculator to determine the 12th power of your answer to part b). Explain why the answer is not 2.

Link the Ideas

You can use the exponent laws to help simplify expressions with rational exponents.

Exponent Law	
Note that a and b are rational or variable bases and m and n are rational exponents.	
Product of Powers	$(a^m)(a^n) = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Power of a Power	$(a^m)^n = a^{mn}$
Power of a Product	$(ab)^m = (a^m)(b^m)$
Power of a Quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
Zero Exponent	$a^0 = 1, a \neq 0$

To simplify expressions with rational exponents, you can use the following principle as well as the exponent laws.

- $a^{-n} = \frac{1}{a^n}, a \neq 0$ $\quad$ $3^{-0.2} = \frac{1}{3^{0.2}}$
- $\frac{1}{a^{-n}} = a^n, a \neq 0$ $\quad$ $\frac{1}{3^{-0.2}} = 3^{0.2}$

Example 1 Multiply or Divide Powers With the Same Base

Write each product or quotient as a power with a single exponent.

a) $\left(5^{\frac{1}{3}}\right)\left(5^{\frac{5}{3}}\right)$ $\quad$ **b)** $(x^5)\left(x^{-\frac{1}{2}}\right)$ $\quad$ **c)** $\frac{3^{-\frac{3}{4}}}{3^{0.25}}$ $\quad$ **d)** $\frac{8^{1.8}}{16^{0.3}}$

Solution

Use the exponent laws for multiplying or dividing powers with the same base and rational exponents.

a) Since the bases are the same, you can add the exponents.

$$\begin{aligned}\left(5^{\frac{1}{3}}\right)\left(5^{\frac{5}{3}}\right) &= 5^{\left(\frac{1}{3}+\frac{5}{3}\right)}\\ &= 5^{\frac{6}{3}}\\ &= 5^2\end{aligned}$$

b) Since the bases are the same, you can add the rational exponents.

$$\left(x^5\right)\left(x^{-\frac{1}{2}}\right) = x^{\left[5 + \left(-\frac{1}{2}\right)\right]}$$
$$= x^{\frac{9}{2}}$$

Convert to $x^{\left[\frac{10}{2} + \left(-\frac{1}{2}\right)\right]}$.

c) Convert the rational exponents so both are fractions or decimal numbers. Then, since the bases are the same, you can subtract the exponents.

$$\frac{3^{-\frac{3}{4}}}{3^{0.25}} = \frac{3^{-0.75}}{3^{0.25}}$$
$$= 3^{-0.75 - 0.25}$$
$$= 3^{-1} \text{ or } \frac{1}{3}$$

d) Convert to the same base. Then, subtract the exponents.

$$\frac{8^{1.8}}{16^{0.3}} = \frac{\left(2^3\right)^{1.8}}{\left(2^4\right)^{0.3}}$$
$$= \frac{2^{5.4}}{2^{1.2}}$$
$$= 2^{4.2}$$

How can you use powers of 2 to convert to the same base?

Your Turn

Write each expression as a power with a single exponent.

a) $(x^{1.5})(x^{3.5})$

b) $\left(p^{\frac{-5}{4}}\right)\left(p^{\frac{1}{2}}\right)$

c) $\dfrac{4^{\frac{1}{2}}}{4^{0.5}}$

d) $\dfrac{1.5^{\frac{4}{3}}}{1.5^{\frac{1}{6}}}$

Example 2 Simplify Powers With Rational Exponents

Write each expression as a power with a single, positive exponent. Then, evaluate where possible.

a) $(4x^3)^{0.5}$

b) $\left[\left(x^3\right)\left(x^{\frac{3}{2}}\right)\right]^{\frac{1}{2}}$

c) $\left(\dfrac{3^4}{16}\right)^{-0.75}$

Solution

a) Raise each term to the exponent, then multiply the exponents.

$$(4x^3)^{0.5} = (4^{0.5})(x^3)^{0.5}$$
$$= 2x^{(3)(0.5)}$$
$$= 2x^{1.5} \text{ or } 2x^{\frac{3}{2}}$$

What is the value of $4^{0.5}$?

b) Method 1: Add the Exponents

Since the bases are the same, you can add the exponents. Raise the result to the exponent $\frac{1}{2}$. Then, multiply.

$$\begin{aligned}\left[(x^3)\left(x^{\frac{3}{2}}\right)\right]^{\frac{1}{2}} &= \left(x^{\left(3+\frac{3}{2}\right)}\right)^{\frac{1}{2}}\\ &= \left(x^{\frac{9}{2}}\right)^{\frac{1}{2}}\\ &= x^{\left(\frac{9}{2}\right)\left(\frac{1}{2}\right)}\\ &= x^{\frac{9}{4}}\end{aligned}$$

$$\begin{aligned}3 + \frac{3}{2} &= \frac{6}{2} + \frac{3}{2}\\ &= \frac{9}{2}\end{aligned}$$

Method 2: Apply Power of a Power

Raise each power to the exponent $\frac{1}{2}$. Then, add the exponents of the resulting powers.

$$\begin{aligned}\left[(x^3)\left(x^{\frac{3}{2}}\right)\right]^{\frac{1}{2}} &= \left[(x^3)^{\frac{1}{2}}\right]\left[\left(x^{\frac{3}{2}}\right)^{\frac{1}{2}}\right]\\ &= \left[x^{(3)\left(\frac{1}{2}\right)}\right]\left[x^{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}\right]\\ &= \left(x^{\frac{3}{2}}\right)\left(x^{\frac{3}{4}}\right)\\ &= x^{\left(\frac{6}{4}+\frac{3}{4}\right)}\\ &= x^{\frac{9}{4}}\end{aligned}$$

c) Convert the base to a single fraction with the same exponent. Then, raise the result to the exponent $-\frac{3}{4}$.

$$\begin{aligned}\left(\frac{3^4}{16}\right)^{-0.75} &= \left(\frac{3^4}{2^4}\right)^{-\frac{3}{4}}\\ &= \left[\left(\frac{3}{2}\right)^4\right]^{-\frac{3}{4}}\\ &= \left(\frac{3}{2}\right)^{(4)\left(-\frac{3}{4}\right)}\\ &= \left(\frac{3}{2}\right)^{-3}\\ &= \left(\frac{2}{3}\right)^{3}\\ &= \frac{8}{27}\end{aligned}$$

Why is $\left(\frac{3}{2}\right)^{-3}$ the same as $\left(\frac{2}{3}\right)^{3}$?

Your Turn

Simplify and evaluate where possible.

a) $(27x^6)^{\frac{2}{3}}$ **b)** $\left[\left(t^{\frac{4}{3}}\right)\left(t^{\frac{1}{3}}\right)\right]^9$ **c)** $\left(\frac{x^3}{64}\right)^{-\frac{2}{3}}$

Example 3 Apply Powers With Rational Exponents

Food manufacturers use a beneficial bacterium called *Lactobacillus bulgaricus* to make yoghurt and cheese. The growth of 10 000 bacteria can be modelled using the formula $N = 10\,000(2)^{\frac{h}{42}}$, where N is the number of bacteria after h hours.

a) What does the value 2 in the formula tell you?

b) How many bacteria are present after 42 h?

c) How many more bacteria are present after 2 h?

d) How many bacteria are present after 105 h?

Solution

a) The value 2 indicates that the number of bacteria doubles every 42 h.

b) Substitute the value $h = 42$ into the formula and evaluate.

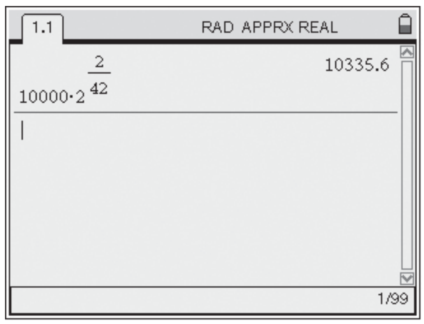
$N = 10\,000(2)^{\frac{42}{42}}$
$N = 10\,000(2)^{1}$
$N = 20\,000$
There are 20 000 bacteria after 42 h.

c) Substitute the value $h = 2$ into the formula and evaluate.

$N = 10\,000(2)^{\frac{2}{42}}$
$N = 10\,000(1.033\,558\ldots)$
$N = 10335.58\ldots$
$10\,335.58\ldots - 10\,000 = 335.58\ldots$

Why do you subtract 10 000?

There are approximately 336 more bacteria after 2 h.



d) Substitute the value $h = 105$ into the formula and evaluate.

$N = 10\,000(2)^{\frac{105}{42}}$
$N = 10\,000(5.656\,854\ldots)$
$N = 56\,568.54\ldots$
There are approximately 56 569 bacteria after 105 h.

Your Turn

Cody invests \$5000 in a fund that increases in value at the rate of 12.6% per year. The bank provides a quarterly update on the value of the investment using the formula $A = 5000(1.126)^{\frac{q}{4}}$, where q represents the number of quarterly periods and A represents the final amount of the investment.

a) What is the relationship between the interest rate of 12.6% and the value 1.126 in the formula?

b) What is the value of the investment after the 3rd quarter?

c) What is the value of the investment after 3 years?

Key Ideas

- You can write a power with a negative exponent as a power with a positive exponent.

$$(-9)^{-1.3} = \frac{1}{(-9)^{1.3}} \qquad \frac{1}{2^{-3.2}} = 2^{3.2}$$

- You can apply the above principle to the exponent laws for rational exponents.

Exponent Law	Example
Note that a and b are rational or variable bases and m and n are rational exponents	
Product of Powers $(a^m)(a^n) = a^{m+n}$	$\left(x^{\frac{3}{5}}\right)\left(x^{\frac{6}{5}}\right) = x^{\frac{3}{5}+\frac{6}{5}}$ $= x^{\frac{9}{5}}$
Quotient of Powers $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{4s^{2.5}}{12s^{0.5}} = \frac{1}{3}s^{(2.5-0.5)}$ $= \frac{1}{3}s^2$ or $\frac{s^2}{3}$
Power of a Power $(a^m)^n = a^{mn}$	$(t^{3.3})^{\frac{1}{3}} = t^{(3.3)\left(\frac{1}{3}\right)}$ $= t^{1.1}$
Power of a Product $(ab)^m = (a^m)(b^m)$	$\left(8x^{\frac{1}{2}}\right)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}\left(x^{\frac{1}{2}}\right)^{\frac{2}{3}}$ $= 4x^{\frac{2}{6}}$ or $4x^{\frac{1}{3}}$ How does expressing 8 as 2^3 help simplify?
Power of a Quotient $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{x^3}{y^6}\right)^{\frac{1}{3}} = \frac{(x^3)^{\frac{1}{3}}}{(y^6)^{\frac{1}{3}}}$ $= \frac{x}{y^2}$
Zero Exponent $a^0 = 1, a \neq 0$	$(-2)^0 = 1$ $-2^0 = -1$

- A power with a rational exponent can be written with the exponent in decimal or fractional form.

$$x^{\frac{3}{5}} = x^{0.6}$$

Check Your Understanding

Practise

1. Use the exponent laws to simplify each expression. Where possible, compute numerical values.

a) $(x^3)\left(x^{\frac{7}{3}}\right)$ **b)** $\left(b^{\frac{1}{5}}\right)\left(b^{\frac{9}{5}}\right)$ **c)** $(a^2)^{\frac{3}{2}}$

d) $(k^{4.8})(k^3)$ **e)** $(16)^{0.25}$ **f)** $\left(\frac{-8a^6}{27}\right)^{\frac{1}{3}}$

g) $\left(2x^{\frac{1}{3}}\right)\left(-4x^{\frac{5}{3}}\right)$ **h)** $(9x^2)^{\frac{3}{2}}$ **i)** $(25x^2)^{0.5}$

2. Use the exponent laws to simplify each expression. Leave your answers with positive exponents.

a) $(x^3)\left(x^{\frac{-2}{3}}\right)$ **b)** $(81^{-0.25})^3$ **c)** $\frac{(m^{-2})^{\frac{2}{3}}}{\left(m^{\frac{1}{2}}\right)^4}$

d) $(9p^2)^{-\frac{1}{2}}\left(p^{-\frac{3}{2}}\right)$ **e)** $\left[\frac{x^{-2}}{(xy)^4}\right]^{1.5}$ **f)** $\left[\frac{4x^{-2}}{9y^{-4}}\right]^{-\frac{5}{2}}$

3. For each of the following, use the exponent laws to help identify a value for p that satisfies the equation.

a) $(x^p)^{\frac{1}{3}} = x^{\frac{2}{3}}$ **b)** $(x^p)\left(x^{\frac{3}{4}}\right) = x^2$

c) $\frac{x^p}{x^{-2}} = x^{\frac{5}{2}}$ **d)** $\left(-3x^{\frac{5}{2}}\right)\left(px^{-\frac{1}{2}}\right) = \frac{-3}{4}x^2$

e) $\left(\frac{9a^{-4}}{25}\right)^p = \frac{3}{5a^2}$ **f)** $(2^{-p})(3^p) = \frac{27}{8}$

4. Evaluate without using a calculator. Leave your answers as rational numbers.

a) $8^{\frac{2}{3}}$ **b)** $16^{\frac{1}{4}}$ **c)** $-27^{\frac{4}{3}}$

d) $\left(3^{\frac{1}{6}}\right)\left(3^{\frac{5}{6}}\right)$ **e)** $\left(\frac{36x^0}{25}\right)^{1.5}$ **f)** $\frac{6^{-2}}{36^{-\frac{1}{2}}}$

5. Evaluate using a calculator. Express your answers to four decimal places, if necessary.

a) $(81^{-0.25})^3$ **b)** $(8^3)(8^{1.2})$ **c)** $\left(\frac{2^5}{5^2}\right)^{-\frac{3}{2}}$

d) $\left(\frac{2^3}{8^2}\right)^{\frac{2}{3}}$ **e)** $\left(\frac{-64}{6^{\frac{1}{2}}}\right)^{\frac{4}{3}}$ **f)** $\frac{\left(2^{\frac{1}{2}}\right)^3}{16}$

6. Whonnock Lake, BC is stocked with rainbow trout. The population grows at a rate of 10% per month. The number of trout stocked is given by the expression $250(1.1)^n$, where n is the number of months since the start of the trout season.

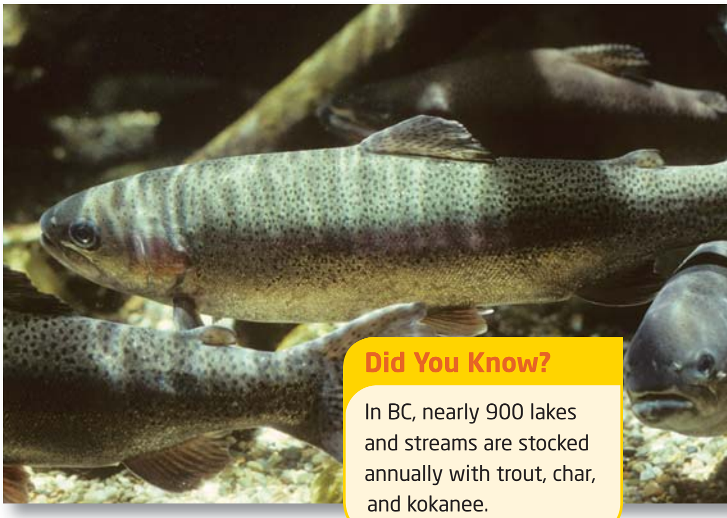
Calculate the number of trout

a) 5 months after the season opens

b) $4\frac{1}{2}$ months after the season opens

c) 2 months before the season opens

d) $3\frac{1}{2}$ months before the season opens



Did You Know?

In BC, nearly 900 lakes and streams are stocked annually with trout, char, and kokanee.

Apply

7. For each solution, find the step where an error was made. What is the correct answer? Compare your corrections with those of a classmate.

a) $\frac{t^{1.2}}{t^{-0.5}} = t^{(1.2 - 0.5)}$
$= t^{0.7}$

b) $(16x^2)^{0.5} = (16^{0.5})(x^2)^{0.5}$
$= 8x^{(2)(0.5)}$
$= 8x$

8. Kelly has been saving money she earned from her paper route for the past two years. She has saved \$1000 to put towards the purchase of a car when she graduates high school. Kelly has two options for investing the money. If she deposits the money into a 3-year term deposit, it earns 1.5% interest per year, but if she deposits the money into a 2-year term deposit, it will earn 2% interest per year. The formula for calculating the value of her investment is $A = 1000(1 + i)^n$, where A is the amount of money at the end of the term, i is the interest rate as a decimal number, and n is the number of years the money is in the term deposit.

a) Which term deposit will give her the most interest?

b) How much more interest does this option pay?

9. From the beginning of 2003 to the beginning of 2007, the population of Manitoba increased at an average annual rate of 0.5%. This situation can be modelled with the equation $P = 1.1619(1.005)^n$, where P is the population, in millions, and n is the number of years since the beginning of 2003.

a) What do you think the number 1.1619 represents?

b) Assuming that the growth rate continues, what will be the population of Manitoba after 15.5 years?

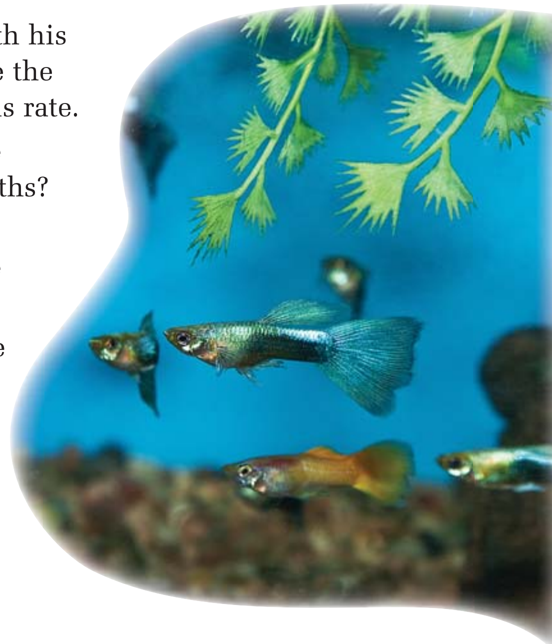
c) Assuming that the growth rate was the same prior to 2003, what was the population of Manitoba at the beginning of 1999?

10. Chris buys six guppies. Every month his guppy population doubles. Assume the population continues to grow at this rate.

a) How many guppies will there be after 1 month? 2 months? 3 months? n months?

b) How many guppies will there be after 6.5 months?

c) Can the fish population continue to grow at this rate? Explain.



11. A mutual fund with an initial value of \$10 000 is decreasing in value at a rate of 12% per year. This situation can be represented by the equation $V = 10\,000(0.88)^n$, where V represents the value of the fund and n the number of years.

a) At this rate, what will be the value of the mutual fund in 5 years and 3 months?

b) If the rate of loss was the same for previous years, what was the value of the fund 3.5 years ago?

12. Martine uses a photographic enlarger that can enlarge a picture to 150% of its previous size. This situation can be modelled by the formula $S = 1.5^t$, where S is the percent increase in the picture size as a decimal number and t is the number of times the enlarger is used.

150% means 1.5 times. ME

a) By how many times is a picture enlarged if the enlarger is used 5 times?

b) How many times would the enlarger need to be used to make a picture at least 25 times as large as the original?

13. Water blocks out sunlight in proportion to its depth. In Qamani'tuaq Lake, NU, $\frac{9}{10}$ of the sunlight reaching the surface of the water can still be seen at a depth of 1 m. This situation can be modelled by the formula $S = 0.9^d$, where S is the fraction of sunlight seen at a depth of d metres. How much sunlight can be seen at a depth of

a) 7.8 m? **b)** 2.75 m?

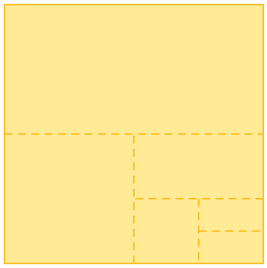
14. Under certain conditions, the temperature, T, in degrees Celsius, of a cooling object can be modelled using the formula $T = 40(10^{-0.1t})$. In this formula, t is time, in minutes. What is the temperature

a) after 10 min? **b)** after 0.25 h?

15. For any planetary system, the orbital radius of a planet, R, in metres, can be predicted using the formula $R = (KT^2)^{\frac{1}{3}}$. In this formula, K is a constant for the system, and T is the orbital period of the planet. The value of K for objects orbiting the sun is $(3.37)(10^{18})$. If it takes Mercury $(7.60)(10^6)$ s to orbit the sun, what is the orbital radius of Mercury, in metres?

Extend

16. When a sheet of paper is folded in half, the area of the paper is reduced by half. This situation can be represented by the equation $A = A_0\left(\frac{1}{2}\right)^f$. In this equation, A_0 represents the starting area of the piece of paper and f the number of consecutive folds. How many folds are needed before the area of the folded paper is less than 1% of the original area? Is this possible? Try it.



17. Julia is a veterinarian who needs to determine the remaining concentration of a particular drug in a horse's bloodstream. She can model the concentration using the formula $C = C_0\left(\frac{1}{2}\right)^{\frac{t}{4}}$, where C is an estimate of the remaining concentration of drug in the bloodstream in milligrams per millilitre of blood, C_0 is the initial concentration, and t is the time in hours that the drug is in the bloodstream. At 10:15 a.m. the concentration of drug in the horse's bloodstream was 40 mg/mL.

a) If only a single dose of the drug is given, what will the approximate concentration of the drug be 6 h later?

b) Julia needs to administer a second dose of the drug when the concentration in the horse's bloodstream is down to 30 mg/mL. Estimate after how many hours this would occur.

Create Connections

18. Describe a problem where rational exponents are used to model a real-life situation. Discuss with a classmate what the rational exponent represents in your problem.

19. Describe at least one common error made when simplifying expressions that include powers with rational exponents. Think of at least one strategy you can use to avoid making the error.

Did You Know?

Johannes Kepler (1571–1630) was a German mathematician and astronomer. He was the first to correctly explain planetary motion.