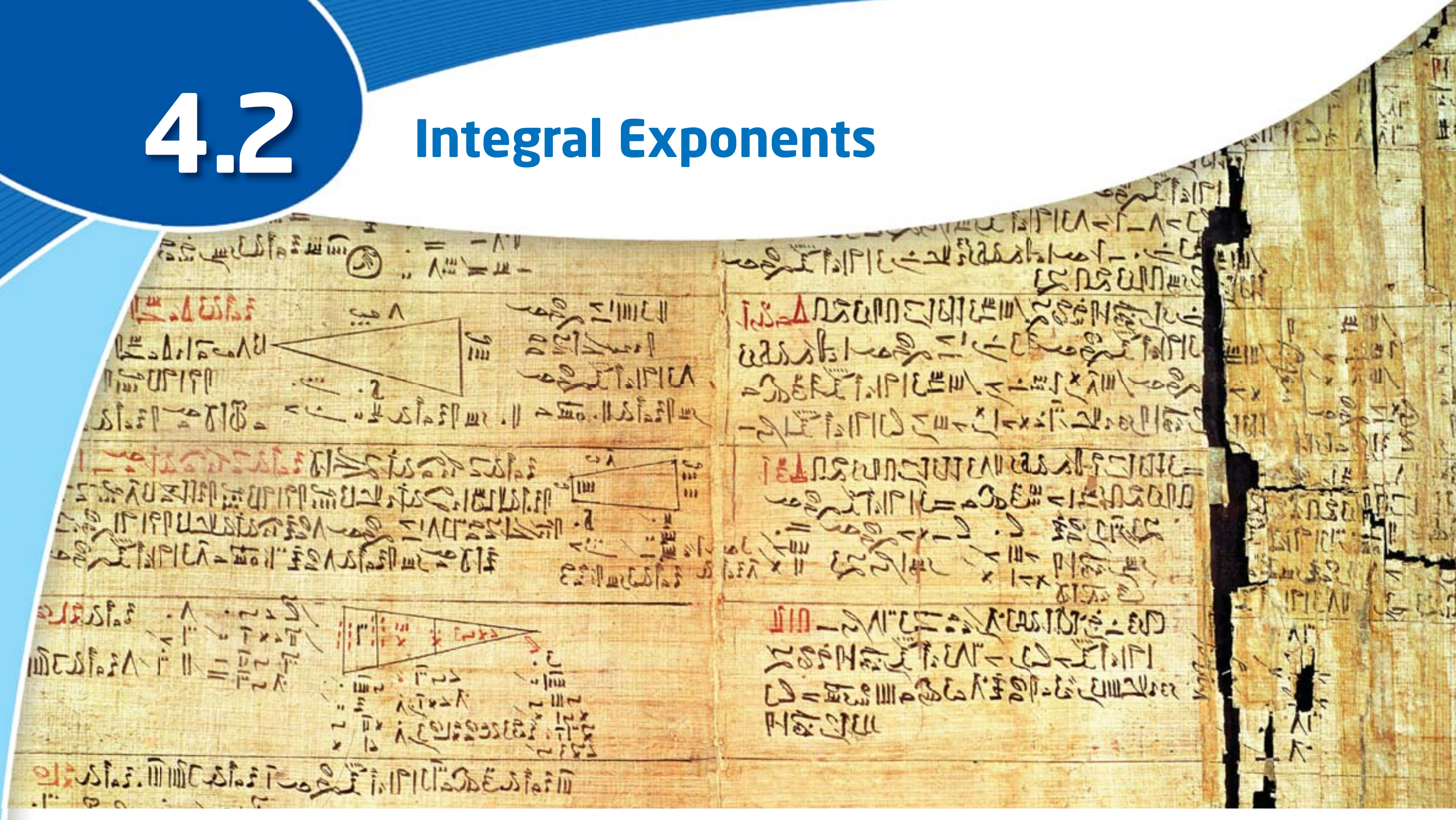
4.2 Integral Exponents



Focus on …

- applying the exponent laws to expressions using rational numbers or variables as bases and integers as exponents
- converting a power with a negative exponent to an equivalent power with a positive exponent
- solving problems that involve powers with integral exponents

The Rhind mathematical papyrus (RMP) is a valuable source of information about ancient Egyptian mathematics. This practical handbook includes problems that illustrate how Egyptians solved problems related to surveying, building, and accounting. The RMP was written in approximately 1650 B.C.E. How do archaeologists know this?

One way to assess the age of organic matter is by using carbon-14 dating. While they are living, all living things absorb radioactive carbon-14. Papyrus is made from papyrus plants. As soon as the papyrus dies, it stops taking in new carbon. The carbon-14 decays at a constant, known rate and is not replaced. Scientists can measure the amount of carbon-14 remaining. They use a formula involving exponents to accurately assess the age of the papyrus.

Is the quantity of carbon-14 increasing or decreasing? Do you think the exponent in the formula would be positive or negative? Why?

Did You Know?

Carbon-14 dating is accurate for dating artifacts up to about 60 000 years old.

Investigate Negative Exponents

Materials
- ruler

1. On a sheet of paper, draw a line 16 cm long and mark it as shown.

2. Mark a point halfway between 0 and 16. Label the point with its value and its equivalent value in exponential form (2^x). Repeat this procedure until you reach a value of 1 cm.

a) How many times did you halve the line segment to reach 1 cm?

b) What do you notice about the exponents as you keep reducing the line segment by half?

3. a) Mark the halfway point between 0 and 1. What fraction does this represent?

b) Using the pattern established in step 2, what is the exponential form of the fraction?

c) Halve the remaining line segment two more times.

4. Use a table to summarize the line segment lengths and the equivalent exponential form in base 2.

5. Reflect and Respond

a) Describe the pattern you observe in the exponents as the distance is halved.

b) Is there a way to rewrite each fraction so that it is expressed as a power with a positive exponent? Try it. Compare this form to the equivalent power with a negative exponent. What is the pattern?

c) Create a general form for writing any power with a negative exponent as an equivalent power with a positive exponent.

6. a) Carbon-14 has a half-life of 5700 years. This means the rate of decay is $\frac{1}{2}$ or 2^{-1} every 5700 years. What fraction of carbon-14 would be present in organic material that is 11 400 years old? 17 100 years old? Express each answer as a power with a negative exponent. Explain how you arrived at your answers.

b) Suggest types of situations when a negative exponent might be used.

Did You Know?

The *half-life* of a radioactive element is the amount of time it takes for half of the atoms in a sample to decay. The half-life of a radioactive element is constant. It does not depend upon the quantity or the amount of time that has gone by.

Link the Ideas

You can use the exponent laws to help simplify expressions with integral exponents.

Exponent Law	
Note that a and b are rational or variable bases and m and n are integral exponents.	
Product of Powers	$(a^m)(a^n) = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Power of a Power	$(a^m)^n = a^{mn}$
Power of a Product	$(ab)^m = (a^m)(b^m)$
Power of a Quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
Zero Exponent	$a^0 = 1, a \neq 0$

To simplify expressions with integral exponents, you can use the following principle as well as the exponent laws.

A power with a negative exponent can be written as a power with a positive exponent.

- $a^{-n} = \frac{1}{a^n}, a \neq 0$ $\quad 2^{-3} = \frac{1}{2^3}$
- $\frac{1}{a^{-n}} = a^n, a \neq 0$ $\quad \frac{1}{2^{-3}} = 2^3$

Example 1 Multiply or Divide Powers With the Same Base

Write each product or quotient as a power with a single exponent.

a) $(5^8)(5^{-3})$

b) $(0.8^{-2})(0.8^{-4})$

c) $\frac{x^5}{x^{-3}}$

d) $\frac{(2x)^3}{(2x)^{-2}}$

Solution

Use the exponent laws for multiplying or dividing powers with the same base and integral exponents.

a) Method 1: Add the Exponents

$$(5^8)(5^{-3}) = 5^{8+(-3)}$$
$$= 5^5$$

How do you know that you can add the exponents?

Method 2: Use Positive Exponents

Convert the power with a negative exponent to one with a positive exponent. Rewrite as a division statement. Then, since the bases are the same, you can subtract the exponents.

$$\begin{aligned}(5^8)(5^{-3}) &= (5^8)\left(\frac{1}{5^3}\right)\\ &= \frac{5^8}{5^3}\\ &= 5^{8-3}\\ &= 5^5\end{aligned}$$

How do you know that you can subtract the exponents?

b) Method 1: Add the Exponents

$$\begin{aligned}(0.8^{-2})(0.8^{-4}) &= 0.8^{-2+(-4)}\\ &= 0.8^{-6}\end{aligned}$$

Method 2: Use Positive Exponents

$$\begin{aligned}(0.8^{-2})(0.8^{-4}) &= \left(\frac{1}{0.8^2}\right)\left(\frac{1}{0.8^4}\right)\\ &= \frac{1}{(0.8^2)(0.8^4)}\\ &= \frac{1}{0.8^{(2+4)}}\\ &= \frac{1}{0.8^6}\end{aligned}$$

Which method do you prefer? Why?

c)

$$\begin{aligned}\frac{x^5}{x^{-3}} &= x^{5-(-3)}\\ &= x^{5+3}\\ &= x^8\end{aligned}$$

What strategy was used?

d) Method 1: Subtract the Exponents

$$\begin{aligned}\frac{(2x)^3}{(2x)^{-2}} &= (2x)^{3-(-2)}\\ &= (2x)^5\end{aligned}$$

Method 2: Use Positive Exponents

$$\begin{aligned}\frac{(2x)^3}{(2x)^{-2}} &= (2x)^3\,(2x)^2\\ &= (2x)^{3+2}\\ &= (2x)^5\end{aligned}$$

Your Turn

Simplify each product or quotient.

a) $(2^{-3})(2^5)$

b) $\frac{7^{-5}}{7^3}$

c) $\frac{(-3.5)^4}{(-3.5)^{-3}}$

d) $\frac{(3y)^2}{(3y)^{-6}}$

Example 2 Powers of Powers

Write each expression as a power with a single, positive exponent. Then, evaluate where possible.

a) $(4^3)^{-2}$ **b)** $[(a^{-2})(a^0)]^{-1}$ **c)** $\left(\frac{2^4}{2^6}\right)^{-3}$ **d)** $\left[\left(\frac{3}{4}\right)^{-2}\left(\frac{3}{4}\right)^{-2}\right]^{-2}$

Solution

a) Multiply the exponents. Then, rewrite as a positive exponent.

$$\begin{aligned}(4^3)^{-2} &= 4^{(3)(-2)}\\ &= 4^{-6}\\ &= \frac{1}{4^6}\\ &= \frac{1}{4096}\end{aligned}$$

b) Since the bases are the same, you can multiply the powers by adding the exponents. Raise the result to the exponent -1. Then, multiply.

$$\begin{aligned}[(a^{-2})(a^0)]^{-1} &= (a^{-2\,+\,0})^{-1}\\ &= (a^{-2})^{-1}\\ &= a^{(-2)(-1)}\\ &= a^2\end{aligned}$$

How could you use your knowledge of the exponent laws for zero exponents to help simplify the original expression?

c) **Method 1: Simplify Within the Brackets**

Since the bases are the same, you can subtract the exponents. Raise the result to the exponent -3. Then, multiply.

$$\begin{aligned}\left(\frac{2^4}{2^6}\right)^{-3} &= [2^{(4\,-\,6)}]^{-3}\\ &= (2^{-2})^{-3}\\ &= 2^{(-2)(-3)}\\ &= 2^6\\ &= 64\end{aligned}$$

Method 2: Raise Each Power to an Exponent

Raise each power to the exponent -3. Then, divide the resulting powers by subtracting the exponents, since they have the same base.

$$\begin{aligned}\left(\frac{2^4}{2^6}\right)^{-3} &= \frac{(2^4)^{-3}}{(2^6)^{-3}}\\ &= \frac{2^{(4)(-3)}}{2^{(6)(-3)}}\\ &= \frac{2^{-12}}{2^{-18}}\\ &= 2^{-12\,-\,(-18)}\\ &= 2^6\\ &= 64\end{aligned}$$

Which method do you prefer? Why?

Did You Know?

John Wallis was a professor of geometry at Oxford University in England in 1655. He was the first to explain the significance of zero and negative exponents. He also introduced the current symbol for infinity, ∞.

d) Add the exponents. Raise the resulting power to the exponent -2.

$$\left[\left(\frac{3}{4}\right)^{-2}\left(\frac{3}{4}\right)^{4}\right]^{-2} = \left[\left(\frac{3}{4}\right)^{-2+4}\right]^{-2}$$
$$= \left[\left(\frac{3}{4}\right)^{2}\right]^{-2}$$
$$= \left(\frac{3}{4}\right)^{(2)(-2)}$$
$$= \left(\frac{3}{4}\right)^{-4}$$
$$= \frac{1}{\left(\frac{3}{4}\right)^{4}}$$
$$= \left(\frac{4}{3}\right)^{4}$$
$$= \frac{256}{81}$$

How do you know that you can add the exponents?

Why is base now $\frac{4}{3}$ instead of the original base of $\frac{3}{4}$?

Is it true in all cases that you can express a rational number with a negative exponent as its reciprocal with a positive exponent? Try it out.

Your Turn

Simplify and evaluate where possible.

a) $[(0.6^3)(0.6^{-3})]^{-5}$ **b)** $[(t^{-4})(t^3)]^{-3}$ **c)** $\left(\frac{x^6}{x^4}\right)^{-2}$ **d)** $\left[\frac{(y^2)^0}{(y)^3}\right]^{-3}$

Example 3 Apply Powers With Integral Exponents

It is estimated that there are 117 billion grasshoppers in an area of 39 000 km^2 of Saskatchewan. Approximately how many grasshoppers are there per square kilometre?

Solution

Method 1: Use Arithmetic

Divide the number of grasshoppers by the total area.

$$\text{grasshoppers per square kilometre} = \frac{117\,000\,000\,000}{39\,000}$$
$$= 3\,000\,000$$

There are approximately 3 000 000 grasshoppers per square kilometre.

Method 2: Use Exponent Rules

Since you cannot enter numbers as large as 117 billion directly into most calculators, rewrite them using exponential form. Then, use the exponent rules to calculate the power of 10.

$$\text{grasshoppers per square kilometre} = \frac{(117)(10^9)}{(39)(10^3)}$$
$$= (3)(10^{(9-3)})$$
$$= (3)(10^6)$$

There are approximately 3 000 000 grasshoppers per square kilometre.

Is it possible to enter numbers expressed using exponential form directly into your calculator? How would doing this help calculate the answer?

Did You Know?

The clear-winged grasshopper is a pest of grasses and cereal grain crops. These insects can completely destroy barley and wheat fields early in the season. Agricultural field workers conduct grasshopper surveys and produce forecasts to help assess the need for control measures to protect crops.

Your Turn

Manitoba Agriculture, Food and Rural Initiatives staff conducted a grasshopper count. In one 25-km^2 area, there were 401 000 000 grasshoppers. Use the following table to assess the degree of grasshopper infestation in this area.

Grasshopper Density
0–4 per square metre = very light
4–8 per square metre = light
8–12 per square metre = moderate
12–24 per square metre = severe
24 per square metre = very severe

Key Ideas

- A power with a negative exponent can be written as a power with a positive exponent.

$$3^{-4} = \frac{1}{3^4} \qquad \left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \left(\frac{3}{2}\right)^2 \qquad \frac{1}{2^{-5}} = 2^5$$

- You can apply the above principle to the exponent laws.

Exponent Law	Example
Note that a and b are rational or variable bases and m and n are integral exponents.	
Product of Powers $(a^m)(a^n) = a^{m+n}$	$(3^{-2})(3^4) = 3^{-2+4}$ $= 3^2$
Quotient of Powers $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{x^3}{x^{-5}} = x^{3-(-5)}$ $= x^8$
Power of a Power $(a^m)^n = a^{mn}$	$(0.75^4)^{-2} = 0.75^{(4)(-2)}$ $= 0.75^{-8}$ or $\frac{1}{0.75^8}$
Power of a Product $(ab)^m = (a^m)(b^m)$	$(4z)^{-3} = \frac{1}{(4z)^3}$ or $\frac{1}{64z^3}$
Power of a Quotient $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{t}{3}\right)^{-2} = \left(\frac{3}{t}\right)^2$ $= \frac{3^2}{t^2}$ $= \frac{9}{t^2}$
Zero Exponent $a^0 = 1, a \neq 0$	$(4y^2)^0 = 1$ $-(4y^2)^0 = -1$

Check Your Understanding

Practise

1. For each situation, identify when a positive and a negative exponent would be used.
   a) calculating the population growth of a city since 2005 using the expression $150\,000(1.005)^n$
   b) calculating the amount of a radioactive substance remaining from a known sample amount using the expression $25\left(\frac{1}{2}\right)^n$
   c) determining how many bacteria are present in a culture after h hours using the expression $500(2)^h$

2. Write each expression with positive exponents.
   a) b^{-3}
   b) xy^{-4}
   c) $2x^{-2}$
   d) $2x^2y^{-1}$
   e) $-4x^{-5}$
   f) $-2x^{-3}y^{-4}$

3. Daniel was rewriting the expression $\frac{2x^{-3}}{y^5}$ with positive exponents. He quickly recorded $\frac{2}{x^3y^5}$. Is his answer correct? Justify your answer.

4. Simplify each expression. State the answer using positive exponents.
   a) $(4^3)(4^{-5})$
   b) $\frac{3^{-4}}{3^{-2}}$
   c) $\frac{12^3}{12^7}$
   d) $\left(\frac{8^{-1}}{8^0}\right)^3$
   e) $(5^4)^{-2}$
   f) $[(3^2)(2^{-5})]^3$
   g) $\left(\frac{5^2}{4^2}\right)^{-1}$
   h) $(3.2^{-2})^{-3}$
   i) $4[(2)^{-1}(2)^{-2}]^{-1}$

5. Simplify each expression by restating it using positive exponents only.
   a) $\frac{1}{s^2t^{-6}}$
   b) $[(h)^7(h)^{-2}]^{-2}$
   c) $\frac{8t}{t^{-3}}$
   d) $(2x^{-4})^3$
   e) $\left(\frac{n^4}{n^{-4}}\right)^{-3}$
   f) $[(xy^4)^{-3}]^{-2}$

6. Simplify, then evaluate. Express your answers to four decimal places, where necessary.
   a) $(0.5^2)^{-3}$
   b) $\left[\left(\frac{2}{3}\right)^3\right]^{-3}$
   c) $[(5)(5^3)]^{-1}$
   d) $\left(\frac{6^4}{6^4}\right)^{-3}$
   e) $\left(\frac{8}{8^3}\right)^{-4}$
   f) $\left[\left(\frac{3}{4}\right)^{-4} \div \left(\frac{3}{4}\right)^2\right]^{-1}$

7. A mountain pine beetle population can double every year if conditions are ideal. Assume the forest in Jasper National Park, AB, has a population of 20 000 beetles. The formula $P = 20\ 000(2)^n$ can model the population, P, after n years.

   a) How many beetles were there in the forest four years ago? eight years ago?

   b) If the conditions remain ideal, how many beetles will there be two years from now?

8. French-language publishing sales in Canada increased by a growth rate of 1.05 per year from 1996 to 2000. There were sales of \$300 000 in 1996. The formula $S = 300\ 000(1.05)^n$ models the sales, S, after n years. Assume that the growth rate stays constant. What would be the projected sales for 2010?

Apply

9. The bacterium *Escherichia coli* is commonly found in the human intestine. A single bacterium has a width of 10^{-3} mm. The head of a pin has a diameter of 1 mm. How many *Escherichia coli* bacteria can fit across the diameter of a pin?

10. A culture of bacteria in a lab contains 2000 bacterium cells. The number of cells doubles every day. This relationship can be modelled by the equation $N = 2000(2)^t$, where N is the estimated number of bacteria cells and t is the time in days.

    a) How many cells were present for each amount of time?
       i) after two days
       ii) after one week
       iii) two days ago

    b) What does $t = 0$ indicate?

11. The Great Galaxy in Andromeda is about 2 200 000 light years from Earth. Light travels 5 900 000 000 000 miles in a year. How many miles is the Great Galaxy in Andromeda from Earth?

12. A red blood cell is about 0.0025 mm in diameter. How large would it appear if it were magnified 10^8 times?

13. Wildlife biologists are tracking the whooping crane population growth at Wood Buffalo National Park, AB. The crane population increased by a growth rate of 7.3% per year from 2002 to 2008. There were 174 whooping cranes in 2002. The rate of growth can be modelled using the formula $P = 174(1.073)^n$, where P is the estimated population and n is the number of years. If conditions remain constant, what is the projected crane population

a) in 2014?

b) in 2011?

Did You Know?

The whooping crane is an endangered species. One of the three wild populations of whooping cranes in North America summers at Wood Buffalo National Park in Alberta. It winters in Texas. Projects are ongoing to help the wild population recover.

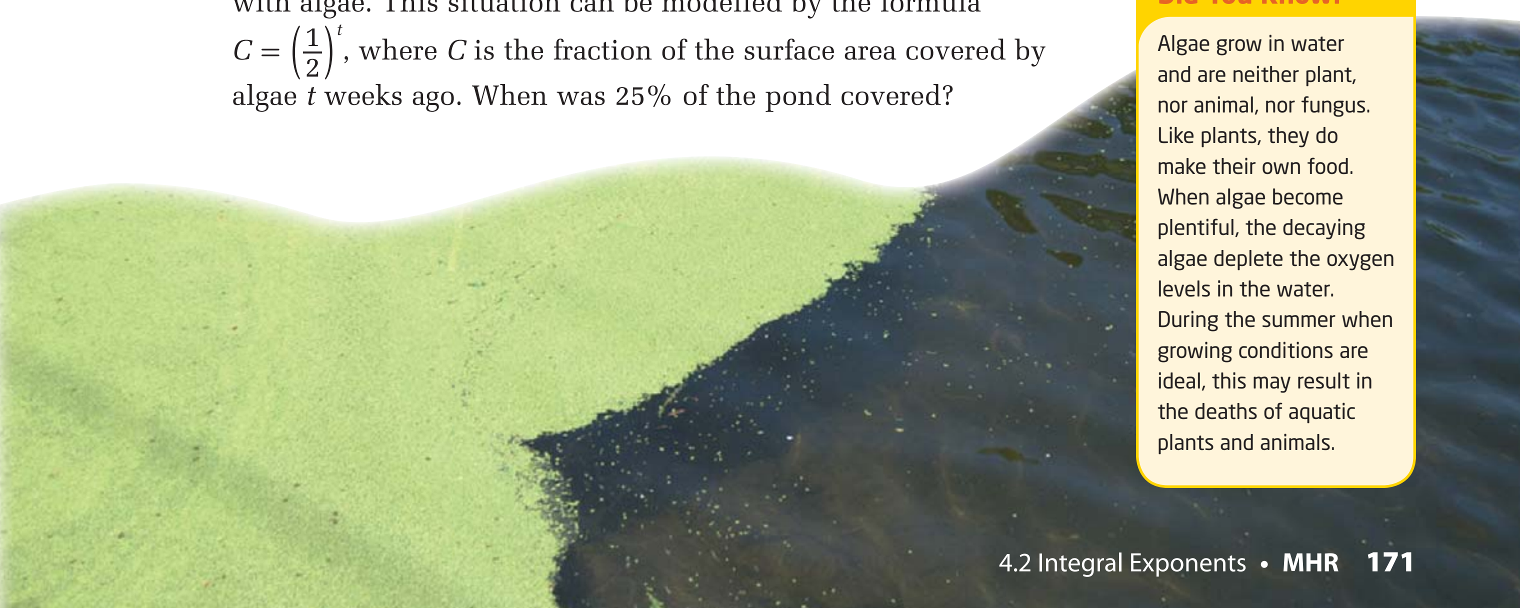
14. There are approximately $(3.2)(10^{18})$ atoms in 1 mg of lead. How many atoms are there in a kilogram of lead? Hint: 1 kg = 10^6 mg.

15. Over time, all rechargeable batteries lose their charge, even when not in use. A 12-volt nickel-metal hydride (NiMH) battery, commonly used in power tools, will lose approximately 30% of its charge every month if not recharged. This situation can be modelled by the formula $V = 12(0.70^m)$, where V is the estimated voltage of the battery in volts, and m is the number of months the battery is not used. What is the estimated voltage of an unused battery after 3 months? Assume the battery was initially fully charged.

16. The fraction of the surface area of a pond covered by algae cells doubles every week. Today, the pond surface is fully covered with algae. This situation can be modelled by the formula $C = \left(\frac{1}{2}\right)^t$, where C is the fraction of the surface area covered by algae t weeks ago. When was 25% of the pond covered?

Did You Know?

Algae grow in water and are neither plant, nor animal, nor fungus. Like plants, they do make their own food. When algae become plentiful, the decaying algae deplete the oxygen levels in the water. During the summer when growing conditions are ideal, this may result in the deaths of aquatic plants and animals.



17. Abby, Kevin, and Caleigh are organizing a 12-hour Famine to raise money to help children in developing countries. Participants are to collect pledges in one of two ways. They can ask for a flat rate of \$30 per pledge or they can ask for a pledge of \$0.01 for the first hour and then, every hour after that, double the pledge from the previous hour. Each hourly pledge using the doubling approach can be modelled by the formula $P = 0.01(2)^h$, where P is the hourly pledge amount and h is the number of hours of participation in the famine.

a) Which pledge approach do you think would raise more money?

b) If the maximum number of hours students can participate in the famine is 12, what is the pledge amount for the last hour?

Did You Know?

A team of Canadian scientists learned previously that crude oil spilled on beaches in Nova Scotia did not break down, due to poor soil conditions. The low nutrient concentrations in the soil limited the growth of natural bacteria. Adding fertilizer to the soil increased the rate of bacterial growth. The scientists applied what they learned to the oil spill in the Arctic, and it worked.

18. Following the 1989 Exxon Valdez oil spill, 100 km of Arctic shoreline was contaminated. Crude oil is made up of thousands of compounds. It takes many different kinds of naturally occurring bacteria to break the oil down. Lab technicians identified and counted the bacteria. They monitored how well the oil was degrading. More bacteria and less oil were signs that the shoreline was recovering. The number of bacteria needed to effectively break down an oil spill is 1 000 000 per millilitre of oil. The bacteria double in number every two days. The starting concentration of bacteria is 1000 bacteria per millimetre. This situation can be modelled by the equation $C = 1000(2^d)$, where C is the estimated concentration of bacteria and d is the number of 2-day periods the bacteria grow. Approximately how long would it take for the bacteria to reach the required concentration?

19. From 2001 to 2006 the population of Lloydminster increased at an average annual rate of 2.52%. This can be modelled using the formula $P = 13\,145(1.0252)^n$, where P is the estimated population and n is the number of years since 2001.

a) What was the population of Lloydminster in 2004?

b) If this rate of increase stays the same, what will the population be in 2012?

Extend

20. Calculate the value of x that makes each statement true.

a) $x^{-4} = \frac{81}{16}$

b) $\left(\frac{1}{3}\right)^x = 81$

c) $\left(\frac{3}{4}\right)^x = \frac{64}{27}$

d) $(-5)^x = \frac{1}{25}$

21. Use your knowledge of equivalent powers to evaluate $\left(\frac{2^5}{8^2}\right)^{-2}$.

22. A fundraising committee plans to donate \$64 000 to six community agencies as follows. The first agency will receive $\frac{1}{2}$. The second agency will receive $\frac{1}{2}$ of what is left. The third agency will receive $\frac{1}{2}$ of what is left, and so on down the line.

a) What fraction of the money will each agency get?

b) How much money will each agency get?

c) Will there be any remaining money after the committee makes these donations? If so, how much?

d) According to the pattern, how many agencies could be supported if no agency is to receive less than \$125?

23. The intensity of light from a stage light decreases exponentially with the thickness of the coloured gels covering it. The intensity, I, in watts per square centimetre, can be calculated using the formula $I = 1200\left(\frac{4}{5}\right)^n$, where n is the number of coloured gels used. What is the intensity of light with

a) no gels? **b)** 2 gels? **c)** 4 gels?

24. The number, N, of radium atoms remaining in a sample that started at 400 atoms can be represented by the equation $N = 400(2)^{\frac{-t}{1600}}$, where t is time, in years.

a) How many atoms are left after 3200 years?

b) What does $t = 0$ represent?

c) What do negative values of t represent?

Create Connections

25. Use a pattern of your choice. Describe the relationship between a negative exponent and its equivalent form with a positive exponent.

26. Describe a problem where negative exponents are used to model a real-life situation. What does the negative exponent represent in your problem?

27. a) Which power is larger: 3^5 or 3^4? Explain how you know. What can you conclude about comparing powers with the same base?

b) Develop an example that shows how to compare powers with the same exponents and different bases. What can you conclude about comparing powers with the same exponents?

c) Which of the following powers is the greatest? How do you know? Arrange the powers from least to greatest.

2^{666} 3^{555} 4^{444} 5^{333} 6^{222}